Faculty Of Graduate Studies

Master Program Of Mathematics

Homogeneous Difference Equations

Prepared By:

Marwa Qadah

Supervise By:

Dr.Marwan Aloqeili

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FACULTY OF GRADUATE STUDIES

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" This Thesis was submitted in partial fulfillment of the requirements for the Master's Degree in Mathematics from the Faculty of Graduate studies at Birzeit University, Palestine."

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This Thesis was successfully defended on June 21,2022

and approved by

Committee members:

Dr. Marwan Aloqeili (Head Of Committee) ______________

Dr. Ala Talahmeh (Internal Examiner) ______________

Dr. Muna Abu Alhalaweh (Internal Examiner) ______________

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الإهداء

إلى جنة دنياي أمي وأبي
إلى من هم سندي في حياتي إخوتي و أخواتي
إلى من لا تطيب الحياة إلا بها صديقة العمر سحر قنديل
إلى من هو خلف الستار
إلى من دعمني وأوصلني إلى بر الأمان الدكتور مروان العقيلي
والشكر موصول إلى لجنة النقاش الدكتور علاء تلاحمة والدكتورة منى أبو الحلاوة
وكل الشكر لمن علمني حرفاً في جامعتي بيرزيت
إلى المعلم الأول سيدنا محمد صلى الله عليه وسلم
كله خالصٌ لوجه الله

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Declaration

I certify that this thesis, submitted for the degree of Master of Science to the department of Mathematics in Birzeit University, is of my own research except where otherwise acknowledged, and that this thesis has not been submitted for a higher degree to any other university or institution.

Marwa Qadah Signature.........

Abstract

This thesis aims mainly to study the relation between reduction of order method for multiplicative and additive homogeneous difference equations of degree one and Lie symmetry method. We study the dynamics of solutions of homogeneous difference equations and consider the reduction of order of such equations. Then we study qualitative behavior of solutions of the reduced equation and its connection to the solution of the original equation.

Keywords: Difference equations; Multiplicative homogeneous difference equations of degree one; Additive homogeneous difference equations of degree one; Lie symmetry; Reduced equation; Local stability; Global stability.

الملخص

تهدف هذه الرسالة بشكل أساسي إلى دراسة العلاقة بين طريقة اختزال الرتب لمعادلات الفرق المتجانسة المضاعفة والمضافة من الدرجة الأولى و طريقة تماثل لي. ثم سنقوم بدراسة ديناميكيات حلول معادلات الفرق المتجانسة و النظر في المعادلات المخفضة لها. ثمّ سندرس السلوك النوعي لحلول المعادلات المخفضة و علاقتها بحل المعادلة الأصلية.

الكلمات المفتاحية: معادلات الفرق، معادلات الفرق المتجانسة المضاعفة من الدرجة الأولى، معادلات الفرق المتجانسة المضافة من الدرجة الأولى، تماثل لي، المعادلات المخفضة، الاستقرار المحلي، الاستقرار العالمي.

CONTENTS

LIST OF SYMBOLS

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{R}$	Real numbers
ODE	Ordinary differential equation
$O\Delta E$	Ordinary difference equation
PDE	Partial differential equation
x_n	$x(n)$
$HD1$	Homogeneous ΔE of degree 1
$MH1$	Multiplicative homogeneous of degree 1
$AH1$	Additive homogeneous of degree 1
r_n	Reduced
v_n	Invariant
s_n	Canonical coordinate
$Q(n, x_n)$	Characteristic function of Lie symmetry
LSC	Linearized symmetry condition
I	Infinitesimal generator
T_0	Trivial symmetry

1 INTRODUCTION

Difference equations constitute an area of considerable interest. There has been some renewed interest in solvable difference equations. A frequent situation is that a difference equation is transformed into a linear first order one, which is solvable. Moreover, an analysis shows that many systems are also essentially reduced to an equation. There are methods that reduce the order of the difference equation as Lie symmetry method.

Meada (1987) has shown that difference equations of order *one* can be solved by Lie's method, and he showed that the linearized symmetry condition (LSC) for such difference equation leads to a set of functional equations. Later, Quisple and Sahdevan (1993) were interested in this method and they extended Meada's idea to a higher order difference equations by using a Laurent series expansion about a fixed point. Levi et al. (1997) expanded the linearized symmetry condition as a series in powers of x_n and looked for symmetries that are more general than point symmetries but the expression derived by them was complicated. Hydon (2000) introduced a method for obtaining the Lie symmetries and used it to reduce the order of the ordinary difference equations and to find the solution. He applied this method to second order difference equations. Walaa Yaseen (2018) used Lie symmetries to find the general solution to difference equations of higher orders as order four.

H. Sedaghat focus on reducing the order of a special type of difference equations which are homogeneous difference equations of degree one. H. Sedaghat (2007) showed that every second order homogeneous difference equation of degree one has a semiconjugte factorizations. And as a result, all second order homogeneous difference equation of degree one can be reduced to a system of two first order difference equations.

H. Sedaghat (2009) generlized his results to homogeneous difference equation of degree one of order $k+1$, and showed that every homogeneous difference equation of degree one of order $k+1$ can be reduced to a system of two equtions of orders k and one respectively.

In this Thesis, we study the symmetry analysis for ordinary difference equations. Then by using Lie symmetries, we investigate the exact solutions for first and second difference equations. And we study the reduction of order method of homogeneous difference equation of degree one. Then we use it to find the general solution for some homogeneous difference equation of degree one.

Also, we find a relation between Lie symmetry method and the reduction of order method for homogeneous difference equation of degree one. Moreover, we study the qualitative behaviour for the original and its reduced equation.

This Thesis is organized as follows, in chapter two, we introduce some basic concepts and the general solution for first order linear difference equations. In chapter three, we investigate symmetries in mathematics to introduce Lie symmetry method for solving first and second order difference equations. And we generlize the method for higher order difference equations.

In chapter four, we investigate order reduction theorem for homogeneous difference equations of degree one. In chapter five, we solve a homogeneous difference equation of degree one using Lie symmetry to find the relation between Lie symmetry and reduction of order method. Finally, we use our results to solve a special case of the difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{a x_{n-k} + b x_{n-l}},$$

and we study the qualitative behaviour of the original and its reduced equation.

2 PRELIMINARIES

2.1 Difference equations

In this section, we present what do we mean by a difference equation, order of a difference equation, types of difference equation and homogeneuous difference equation of order k.

Definition 2.1. [11] *A **difference equation** is an equation that expresses a value of a sequence as a function of the other terms in the sequence, that is, it defines a relation recursively.*

Definition 2.2. [1] *The **order of difference equation** is the difference between highest and lowest indices that appear in the equation.*

The difference equation of order k is of the form

$$x_n = f(x_{n-1}, x_{n-2}, \dots, x_{n-k}), \quad n = 0, 1, 2, \dots \quad , \tag{2.1}$$

where f is a function such that $f : \mathbb{R}^k \to \mathbb{R}$ and the initial conditions $x_{-1}, x_{-2}, \dots, x_{-k}$ are all arbitrary real numbers.

Difference equations can be classified into different types according to one or more of the following properties ([2]):

1. **Linear difference equations:** an equation is said to be linear if the function f in Eq.(2.1) is a linear function.

2. **Non-linear difference equations:** an equation is said to be non-linear if the function f in Eq.(2.1) is a non-linear function.

3. **Linear homogeneous difference equations:** a k^{th} order linear homogeneous difference equation is an equation of the form

$$x_{n+k} + P_1(n)x_{n+k-1} + \cdots + P_k(n)x_n = 0,$$

   where $P_k(n) \neq 0, \ \forall n \geq n_0$.

4. **Linear non-homogeneous difference equations:** a k^{th} order linear nonhomogeneous difference equation is an equation of the form

$$x_{n+k} + P_1(n)x_{n+k-1} + \cdots + P_k(n)x_n = g(n),$$

   where $P_k(n) \neq 0, \ \forall n \geq n_0$. The sequence $g(n)$ is called the forcing term.

5. **Autonomous difference equations:** a k^{th} order difference equation is said to be autonomous if it is time-invariant, that is as Eq.(2.1).

6. **Non-autonomous difference equations:** a k^{th} order difference equation is said to be non-autonomous if the function f can be replaced by a new function h of $k+1$ variables such that, $h : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}$, that is

$$x_n = h(n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}).$$

   In this case the equation is time-variant.

7. **Linear difference equations with constant coefficients:** a k^{th} order difference equation is said to be linear with constant coefficients if it is of the form

$$x_{n+k} + P_1 x_{n+k-1} + \cdots + P_k x_n = g(n),$$

where $\forall i = 1, 2, \ldots, k$, $P_i's$ are constants and $P_k \neq 0$.

8. **Linear difference equations with non-constant coefficients:** a k^{th} order difference equation is said to be linear with non-constant coefficients if it is of the form

$$x_{n+k} + P_1(n) x_{n+k-1} + \cdots + P_k(n) x_n = g(n),$$

where $P_k(n) \neq 0, \ \forall n \geq n_0$.

We will give examples about the previous types of difference equations in the upcoming sections.

2.2 Initial Value Problem of a Difference Equations

Definition 2.3. [8] **An initial value problem of a difference equation** is a problem when we know value x_0 at a particular point n_0.

Example 2.4. [12] The function $\eta(n) = 3^n \left(2 + \frac{n(n-1)}{6}\right)$ is a solution for the initial value problem

$$x_{n+1} - 3x_n = 3^n n; \ n \geq 0 \ \text{ and } \ x_0 = 2,$$

since if we substitute $\eta(n)$ into the equation, we get

$$\begin{aligned} 3^{n+1}\left(2 + \frac{n(n+1)}{6}\right) - 3^{n+1}\left(2 + \frac{n(n-1)}{6}\right) &= 3^{n+1}\left(2 + \frac{n^2}{6} + \frac{n}{6} - 2 - \frac{n^2}{6} + \frac{n}{6}\right) \\ &= 3^n n. \end{aligned}$$

Also, we have

$$\eta(0) = 3^0\left(2 + \frac{0(0-1)}{6}\right) = 2 = x_0.$$

It should be clear that for a given difference equation, even if a solution is known to exist, there is no assurance that it will be unique. The solution must be restricted by given a set of initial conditions equal in number to the order of the equation. The following theorem states condition that assure the existence of a unique solution.

Theorem 2.5. [8] Let a k^{th} order difference equation

$$x(n+k) = f(x(n), x(n+1), \cdots, x(n+k-1)); n = 0, 1, 2, \cdots, \tag{2.2}$$

where f is defined for each of its arguments. Then Eq.(2.2) has a unique solution corresponding to each arbitrary selection of the k initial values $x(0) = x_0, x(1) = x_1, \cdots, x(k-1) = x_{k-1}$.

Proof. Suppose that $x(0), x(1), \cdots, x(k-1)$ are given. Then the difference equation with $n = 0$ uniquely specifies $x(k)$. Now $x(k)$ is known, the difference equation with $n = 1$ gives $x(k+1)$. Continue in this way, all x_n for $n \geq k$, can be determined. □

We consider the following initial value problem which is a first order linear homogeneous difference equation with constant coefficients

$$x_{n+1} = ax_n, \qquad n = 0, 1, 2, \cdots \tag{2.3}$$

with $x(0) = x_0$ and a is a constant. By iterations, we get

$$\begin{aligned} x_1 &= ax_0, \\ x_2 &= ax_1 = a^2x_0, \\ x_3 &= ax_2 = a^3x_0, \\ &\vdots \\ x_n &= a^nx_0. \end{aligned} \tag{2.4}$$

To prove Eq.(2.4) is a solution of Eq.(2.3), we proceed as follows

$$x_{n+1} = a^{n+1}x_0 = a(a^nx_0) = ax_n.$$

Now, to generelize Eq.(2.3) to non-homogeneuos difference equations with non-constant coefficients, we get the following theorem.

Theorem 2.6. *[11] Let $a(n)$ and $b(n)$ be real sequences where $n \in \mathbb{N}$. Then the first order linear difference equation*

$$x_{n+1} + a(n)x_n = b(n), \tag{2.5}$$

with initial condition $x_0 = c$, has a unique solution of the form

$$x_n = c\left(\prod_{i=0}^{n-1} -a(i)\right) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1} -a(j)\right)b(i). \tag{2.6}$$

Proof. First, we must show that Eq.(2.6) satisfies Eq.(2.5) and the initial condition. We first write the expression for x_{n+1}

$$x_{n+1} = c\left(\prod_{i=0}^{n} -a(i)\right) + \sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i).$$

We then rewrite the last summation above as follows,

$$\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i) = \prod_{j=n+1}^{n}\left(-a(j)b(n)\right) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n} -a(j)\right)b(i)$$

since

$$\prod_{j=n+1}^{n}\Big(-a(j)\Big) = 1,$$

we get

$$\begin{aligned}\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}-a(j)\right)b(i) &= b(n) + \sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n}-a(j)\right)b(i)\\ &= b(n) - a(n)\left[\sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1}-a(j)\right)b(i)\right].\end{aligned}$$

Using this result we obtain,

$$x_{n+1} = -ca(n)\left(\prod_{i=0}^{n-1}-a(i)\right) + b(n) - a(n)\left(\sum_{i=0}^{n-1}\left[\prod_{j=i+1}^{n-1}-a(j)\right]b(i)\right),$$

which implies

$$x_{n+1} = -a(n)x_n + b(n).$$

Thus, we have shown that x_n is a solution. Finally we must prove uniqueness. Assume that we have two solutions x_n and $\hat{x}_n$, both satisfy Eq.(2.5) and the initial condition. Now, consider the set $\{n \in \mathbb{N}; x_n \neq \hat{x}_n\}$. Let n_0 be the smallest integer in this set. We must have $n_0 \geq 1$, since $x_0 = \hat{x}_0$. By the definition of n_0 we have $x_{n_0-1} = \hat{x}_{n_0-1}$ and then

$$x_{n_0} = a(n_0-1)x_{n_0-1} + b(n_0-1) = a(n_0-1)\hat{x}_{n_0-1} + b(n_0-1) = \hat{x}_{n_0},$$

which is a contradiction. Thus we must have $n_0 = 0$. But $x_0 = \hat{x}_0 = c$ since the two equations satisfy the same initial condition. It follows that the solution is unique. □

3 SYMMETRIES IN MATHEMATICS

Symmetry exists in many places of our life, it takes a big part in geometry, also in other branches of mathematics as calculus, integration, linear algebra, abstract algebra, probability, differential equations and difference equations. Symmetry is a type of invariance: the property that a mathematical object remains unchanged under set of operations or transformations. In this chapter, we review transformation, symmetry in general, symmerty in geometry, symmetry in calculus, symmetry in differential equations and a one parameter local Lie group to have a Lie Symmetry.

Definition 3.1. [12] Transformtion or a mapping of a region A into a region B is a rule that assigns to each point $a \in A$ a unique point $b \in B$.

Definition 3.2. [12] A transformtion is a symmetry if it satisfies the following properties:

- The transformation preserves the structure.
- The transformation is a diffeomorphism, that is a smooth invertible

mapping whose inverse is also smooth.

- The transformation maps the object to itself.

Definition 3.3. [12] Trivial symmetry is the transformation that maps each point to itself.

Symmetry in Geometry: the types of symmetry considered in basic geometry include reflection symmetry, rotation symmetry, translation symmetry.
Symmetry in Calculus: in even and odd functions.

Definition 3.4. Let $f(x)$ be real-valued function of a real variabel. Then f is even, if $f(x) = f(-x)$, $\forall x \in \text{Domain}(f)$.

Geometrically speaking, the graph of an even function is symmetric with respect to the y-axis, meaning that its graph remains unchanged after reflection about the y-axis.
Examples of even functions include x^2, $x^4 - 1$, $\cos x$, and $\cosh x$.
Symmetries of Differential Equations

Definition 3.5. A symmetry of differential equation is a transformation that leaves the differential equation invariant.

Knowledge of such symmetries may help to solve the differential equation.

Definition 3.6. A line symmetry of a system of differential equations is a continuous symmetry of the system of differential equations.

Knowledge of a line symmetry can be used to simplify an ordinary differential equation through reduction of order. For ordinary differential equation, knowledge of an appropriate set of Lie symmetries allows one to explicitly

calculate a set of first integrals, yielding a complete solution without integration.
Symmetries may be found by solving a related set of ordinary differntial equations. Solving these equations is often much simpler than solving original differential equations.

3.0.1 Symmetries Of Difference Equations

In this section, we apply the transformation for difference equations to have a symmetry, and define a one parameter local Lie group.

Definition 3.7. [12] *A transformation of a difference equation is a symmetry if every solution of the transformed equation is a solution of the original equation and vice versa.*

The following example illustrates the above definition.

Example 3.8. [12] The map $T_h : x_n \to \hat{x}_n = hx_n$, $\forall h \in \mathbb{R} - \{0\}$ is a transformation of the linear homogeneuous difference equation of order k:

$$h_k(n)x_{n+k} + h_{k-1}(n)x_{n+k-1} + \cdots + h_0(n)x_n = 0. \tag{3.1}$$

The transformation T_h means that we change each variable x_{n+i}, $i = 0, 1, \ldots, k$ in Eq.(3.1) by $hx_{n+i} = \hat{x}_{n+i}$, $i = 0, 1, \ldots, k$ or we can say that we multiply Eq.(3.1) by a nonzero constant h as follows to have a transformed equation:

$$\begin{aligned} h_k(n)hx_{n+k} + h_{k-1}(n)hx_{n+k-1} + \cdots + h_0(n)hx_n &= 0 \\ h_k(n)\hat{x}_{n+k} + h_{k-1}(n)\hat{x}_{n+k-1} + \cdots + h_0(n)\hat{x}_n &= 0 \end{aligned} \tag{3.2}$$

Now to prove that the transformation T_h is a symmetry, we need to prove that each solution of the transformed Eq.(3.2) is a solution of the original Eq.(3.1) and vice versa.

If $x_1(n), x_2(n), \ldots, x_k(n)$ are linearly independent solutions of Eq.(3.1), then the general solution of Eq.(3.1) is a linear combination of $x_1(n), x_2(n), \ldots, x_k(n)$ $x_n = \sum_{i=1}^{k} m_i x_i(n)$, where m_i are real constants. Need to show $\hat{x}_n$ is a linear combimation of $x_1(n), x_2(n), \ldots, x_k(n)$.
The transformation T_h maps the solution x_n to $\hat{x}_n$ as below

$$\begin{aligned} \hat{x}_n &= hx_n \\ &= h\sum_{i=1}^{k} m_i x_i(n) \\ &= \sum_{i=1}^{k} hm_i x_i(n) \\ &= \sum_{i=1}^{k} \hat{m}_i x_i(n) \quad , \hat{m}_i = hm_i, i = 1, 2, \ldots, k. \end{aligned}$$

Therefore $\hat{x}_n$ is also a solution of Eq.(3.1) since we can write it as a linear combination of $x_1(n), x_2(n), \ldots, x_k(n)$.
Conversely, if $\hat{x}_1(n), \hat{x}_2(n), \ldots, \hat{x}_k(n)$ are linearly independent solutions of Eq.(3.2), then the general solution of Eq.(3.2) is $\hat{x}_n = \sum_{i=1}^{k} r_i \hat{x}_i(n)$, where r_i are real constants.

The inverse transformation is $T_h^{-1} : \hat{x}_n \to h^{-1}x_n$ (since from the second property of the definition a Transformation to be a symmetry thats say, the transformation is a diffomorphism, that is a smooth invertible mapping whose inverse is also smooth). So, T_h^{-1} maps the solution $\hat{x}_n$ to x_n as follows

$$\begin{aligned} x_n &= \frac{1}{h}\hat{x}_n \\ &= \frac{1}{h}\sum_{i=1}^{k} r_i \hat{x}_i(n) \\ &= \sum_{i=1}^{k} \frac{r_i}{h}\hat{x}_i(n) \\ &= \sum_{i=1}^{k} \hat{r}_i \hat{x}_i(n) \quad , \hat{r}_i = \frac{r_i}{h}, i = 1, 2, \ldots, k. \end{aligned}$$

Hence, x_n is also a solution of Eq.(3.2) because we can write it as a linear combination of $\hat{x}_1(n), \hat{x}_2(n), \ldots, \hat{x}_k(n)$. Thus T_h is a symmetry $\forall$ h $\in \mathbb{R} - \{0\}$. The proof is complete.

Note that x_n and $\hat{x}_n$ are two solutions of the linear homogeneous difference equation Eq.(3.1) because $\hat{x}_n$ is a multiple of x_n.

Theorem 3.9. [12] *Consider the set of transformations $\mathbb{G} = \{T_h, h \in \mathbb{R} - \{0\}\}$. Then $\mathbb{G}$ is a group under the composition $T_h T_l = T_{hl}$, for all h,l $\in \mathbb{R} - \{0\}$.*

Proof. :

- $\mathbb{G}$ is closed by definition.
- The identity transformation is $T_1 : x_n \to x_n$.
- The inverse transformation of T_h^{-1} is $T_{h^{-1}} = T_{\frac{1}{h}}$.

- $(T_hT_l)T_o = T_{hl}T_o = T_{(hl)o} = T_{h(lo)} = T_hT_{lo} = T_h(T_lT_o)$ so the associvity property is preserved.

□

Definition 3.10. [12] *Consider the following point transformation*

$$T_h : x \to \hat{x}(x; h), \quad h \in (h_1, h_2) \quad where \quad h_1 < 0 \quad and \quad h_2 > 0.$$

Then T_h is a one parameter local Lie group if the following conditions are satisfied:

1. *T_0 is the identity map, that is $\hat{x} = x$, where $h = 0$*
2. *$T_hT_l = T_{h+l}$, for all h, l sufficiently close to zero.*
3. *Each $\hat{x}$ can be represented by a Taylor series in h, such that*

$$\hat{x}(x; h) = x + h\psi(x) + O(h^2).$$

Some notes for the above definition:

- The above transformation is called a point transformation since the transformed point $\hat{x}$ only depends on the point x.
- A local Lie group is a group if it satisfies the group axioms.
- In general, a one parameter local Lie group will depend only on n and x_n.
- The inverse point transformation $T_h^{-1} = T_{h^{-1}}$, where $| h |$ is so small.

3.1 Lie Symmetry Of Difference Equations

In this section, we review Lie symmetry method for solving difference equations and then, we use it to solve a given first and second order difference equations. Finally, we generalize the method for higher order difference equations.

Definition 3.11. [12] *Let the transformation T_h be a symmetry and a one parameter local Lie group. Then it's called a* ***Lie symmetry****.*

In the following example, we explain Lie symmetry for a first order difference equation.

Example 3.12. [6] Consider the first order difference equation:

$$x_{n+1} - x_n = 0. \tag{3.3}$$

and the transformation

$$T_\alpha : (n, x_n) \to (\hat{n}, \hat{x}_n) = (n, x_n + \alpha); \quad \alpha \in \mathbb{R} \tag{3.4}$$

Then, T_α is a Lie symmetry.

The transformation T_α is a symmetry for Eq.(3.3) since the solution of Eq.(3.3) is $x_n = x_0$. Every transformation with $\alpha \neq 0$ maps each solution, $x_n = x_0$ to $\hat{x}_n = x_0 + \alpha$, which can be written as $\hat{x}_n = c$; $c = x_0 + \alpha$.

And T_α is a one parameter local Lie group, since

1. T_0 is the identity map

$$T_0 : (n, x_n) \to (\hat{n}, \hat{x}_n) = (n, x_n),$$

2. $T_\alpha T_\beta = T_{\alpha+\beta}$ since

$$T_\beta : (n, x_n) \to (n, x_n + \beta),$$

which implies that

$$T_\alpha T_\beta : (n, x_n + \beta) \to (n, x_n + \beta + \alpha).$$

And

$$T_{\alpha+\beta} = (n, x_n) \to (n, x_n + \alpha + \beta).$$

Thus,

$$T_\alpha T_\beta = T_{\alpha+\beta}.$$

3. Each $\hat{x}_n$ can be represented as a Taylor series in α. Such that

$$\hat{n} = n, \qquad \hat{x}_n = x_n + \alpha Q(n, x_n) + O(\alpha^2), \tag{3.5}$$

where $Q(n, x_n)$ is a function of n and x_n that depends on the difference equation and is called a **characteristic of the local Lie group**. In the previous example, the characteristic $Q(n, x_n)$ is 1. Since $\hat{x}_n = x_n + \alpha$ by comparing it with Eq.(3.5), we find that $Q(n, x_n) = 1$.

Remark 3.13. For Lie symmetries, we consider $\hat{n} = n$, that is, we leave n unchanged because n is a discrete variable, that can not be changed by a small amount.

Recall that, we will restrict our attention to Lie symmetries for which $\hat{x}_n$ depends only on n and x_n which are called Lie point symmetries and take the form:

$$\hat{n} = n, \qquad \hat{x}_n = x_n + \alpha Q(n, x_n) + O(\alpha^2).$$

If we replace n by $n + p$ in Eq.(3.5), that's we do a shift, we get

$$\hat{x}_{n+p} = x_{n+p} + \alpha Q(n + p, x_{n+p}) + O(\alpha^2),$$

which is called a prolongation formula for Lie point symmetries.

Now our aim is to use symmetries to obtain exact solutions for difference equations. Because of that, we introduce the change of variables. To consider the effect of changing variables from (n, x_n) to (n, s_n), and as Eq.(3.5) is a symmetry for each α sufficiently close to *zero*, we can apply Taylor's theorem about $\alpha = 0$, to obtain

$$\begin{aligned}
\hat{s}_n &= s(\hat{n}, \hat{x}_n) \\
&= s(n, \hat{x}_n) \\
&= s(n, x_n + \alpha Q(n, x_n) + O(\alpha^2)) \qquad \text{Now apply Taylor's theorem about } \alpha = 0 \\
&= s(n, x_n + \alpha Q(n, x_n)) \mid_{\alpha=0} + (\alpha - 0)\frac{ds}{d\alpha} \mid_{\alpha=0} + O(\alpha^2) \\
&= s(n, x_n) + \alpha(\frac{ds}{d\hat{x}_n})(\frac{d\hat{x}_n}{d\alpha}) \mid_{\alpha=0} + O(\alpha^2) \\
&= s(n, x_n) + \alpha s'(n, x_n)Q(n, x_n) + O(\alpha^2). \qquad (3.6)
\end{aligned}$$

If we denote the characteristic function with respect to (n, s_n) by $\hat{Q}(n, s_n)$ then we get by Taylor series

$$\hat{s}_n = s_n + \alpha\hat{Q}(n, s_n) + O(\alpha^2)$$

Comparing the last equation with Eq.(3.6) we get

$$\hat{Q}(n, s_n) = s'(n, x_n)Q(n, x_n). \qquad (3.7)$$

The coordinate s_n is called the canonical coordinate.
Note that $\frac{d\hat{x}_n}{d\alpha} = Q(n, x_n)$, since $\hat{x}_n = x_n + \alpha Q(n, x_n) + O(\alpha^2)$.

In the following example, we will illustrate the effect of changing coordinates introduced above.

Example 3.14. [6] Consider the change of coordinates from (n, x_n) to (n, s_n), and symmetries for s_n,

$$(\hat{n}, \hat{s}_n) = (n, s_n + \alpha), \qquad \alpha \in \mathbb{R}.$$

Then the characteristic with respect to (n, s_n) is $\hat{Q}(n, s_n) = 1$, so by Eq.(3.7),

$$s'(n, x_n)Q(n, x_n) = 1,$$

which implies that

$$s(n, x_n) = \int \frac{dx_n}{Q(n, x_n)}. \tag{3.8}$$

Now, as an example if $Q(n, x_n) = x_n - 1$, then the canonical coordinate according to Eq.(3.8) is

$$s(n, x_n) = \int \frac{dx_n}{x_n - 1} = \ln|x_n - 1| = \begin{cases} \ln(x_n - 1), & x_n > 1 \\ \ln(1 - x_n), & x_n < 1 \end{cases}$$

In this example, the map from x_n to s_n isn't injective for all $\mathbb{R}$; it can't be inverted from s_n to x_n except if we specify whether x_n is greater or less than 1.

But we are interested in injective maps, to have the exact solution x_n in explicit formula. As we will see in the following sections.

3.1.1 Lie Symmetries for 1^{st} O∆E

In this section, our purpose is to solve a given first order difference equation

$$x_{n+1} = f_n(x_n) = f(n, x_n), \tag{3.9}$$

by Lie symmetries. Recall that, for any transformation of a difference equation to be a symmetry, the set of solutions must be mapped into itself. So the symmetry condition of Eq.(3.9) must be satisfied

$$\hat{x}_{n+1} = f(\hat{n}, \hat{x}_n) \quad \text{when} \quad x_{n+1} = f(n, x_n). \tag{3.10}$$

From the symmetry condition Eq.(3.10), we get

$$\begin{aligned} \hat{f}(n, x_n) &\equiv f(\hat{n}, \hat{x}_n) \\ &= f(n, x_n + \alpha Q(n, x_n) + O(\alpha^2)) \\ &= f(n, x_n) + \alpha f'(n, x_n) Q(n, x_n) + O(\alpha^2). \end{aligned} \tag{3.11}$$

Also, we have from Taylor series

$$\hat{f}(n, x_n) = \hat{x}_{n+1} = x_{n+1} + \alpha Q(n+1, x_{n+1}) + O(\alpha^2). \tag{3.12}$$

So, by comparing Eq.(3.11) with Eq.(3.12) we have

$$Q(n+1, x_{n+1}) = f'(n, x_n) Q(n, x_n). \tag{3.13}$$

This is called the linearized symmetry condition (LSC) for the given Eq. (3.9).

The linearized symmetry condition in Eq.(3.13) is a linear functional equation which is difficult to solve.

The following example illustrates how to find (LSC) for a given first order difference equation

Example 3.15. [6] Consider the equation

$$x_{n+1} - x_n = 0,$$

and $f'(n, x_n) = 1$ since $x_{n+1} = f(n, x_n) = x_n$. So the linearized symmetry condition is

$$Q(n+1, x_{n+1}) = Q(n, x_n).$$

Since $x_{n+1} = x_n$,

$$Q(n+1, x_n) = Q(n, x_n).$$

This condition has the general solution

$$Q(n, x_n) = f(x_n),$$

where f is an arbitrary function. Since Q for n and $n+1$ is the same, therefore, Q is a function that depends only on x_n.

For the above example and others, we can find the general solution if we can solve the (LSC), which is a functional equation, but not all functional equations can be solved. Because of that, for first order difference equations we use a practical approach that depends on assuming a trial solution for the characteristic function in the following form

$$Q(n, x_n) = a(n)x_n^2 + b(n)x_n + c(n), \tag{3.14}$$

where $a(n)$, $b(n)$ and $c(n)$ are functions of n. We can find them after substituting Eq.(3.14) in Eq.(3.13) and comparing the powers of x_n.

Example 3.16. [6] Find the characteristic function for the following first order difference equation

$$x_{n+1} = \frac{x_n}{1 + nx_n}, \quad n \geq 1. \tag{3.15}$$

Solution: we need to find f' and then substitute it in LSC as follows:
$f(n, x_n) = \frac{x_n}{1+nx_n}$. So, $f'(n, x_n) = \frac{1}{(1+nx_n)^2}$. Then the LSC is

$$Q(n+1, x_{n+1}) = \frac{1}{(1+nx_n)^2} Q(n, x_n).$$

Now we use the trial solution (3.14), to get

$$a(n+1)x_{n+1}^2 + b(n+1)x_{n+1} + c(n+1) = \frac{1}{(1+nx_n)^2}\Bigg(a(n)x_n^2 + b(n)x_n + c(n)\Bigg).$$

Now substitue $x_{n+1} = \frac{x_n}{1+nx_n}$ to get,

$$a(n+1)\frac{x_n^2}{(1+nx_n)^2} + b(n+1)\frac{x_n}{1+nx_n} + c(n+1) = \frac{1}{(1+nx_n)^2}\Bigg(a(n)x_n^2 + b(n)x_n + c(n)\Bigg), \quad (3.16)$$

After multiplying Eq.(4.20) by $(1+nx_n)^2$, we get

$$a(n+1)x_n^2+b(n+1)x_n(1+nx_n)+c(n+1)(1+nx_n)^2 = a(n)x_n^2+b(n)x_n+c(n).$$

Therefore,

$$a(n+1)x_n^2+b(n+1)x_n+nb(n+1)x_n^2+c(n+1)+2nc(n+1)x_n+n^2c(n+1)x_n^2 = a(n)x_n^2+b(n)x_n+c(n). \quad (3.17)$$

Rearrange the above equation as follows:

$$\Bigg(a(n+1)+nb(n+1)+n^2c(n+1)\Bigg)x_n^2+\Bigg(b(n+1)+2nc(n+1)\Bigg)x_n+c(n+1) = a(n)x_n^2+b(n)x_n+c(n). \quad (3.18)$$

Comparing the powers of x_n, we get the following system:

$$a(n+1)+nb(n+1)+n^2c(n+1) = a(n), \quad (3.19)$$

$$b(n+1)+2nc(n+1) = b(n), \quad (3.20)$$

$$c(n+1) = c(n). \quad (3.21)$$

We solve the above system by using backward substitution, starting with Eq.(3.21), which is a first order linear homogeneous difference equation whose solution is

$$c(n) = \alpha, \qquad \alpha \in \mathbb{R}$$

Then, substitute $c(n) = \alpha$ in Eq. (3.20) to get also a first order difference equation

$$b(n+1) - b(n) = -2n\alpha,$$

whose solution is

$$b(n) = \beta - \sum_{i=0}^{n-1}(2\alpha i).$$

So

$$b(n) = \beta - \alpha n(n-1), \quad \alpha, \beta \in \mathbb{R}.$$

The last step is to substitute $b(n)$ and $c(n)$ in Eq.(3.19), to get

$$a(n+1) - a(n) = -n\beta + n^3\alpha$$

Which is also a first order linear non-homogeneous difference equation, whose general solution is

$$\begin{aligned} a(n) &= \gamma - \sum_{i=0}^{n-1}(\beta i) + \sum_{i=0}^{n-1}(\alpha i^3) \\ &= \gamma - \beta\frac{n(n-1)}{2} + \alpha\frac{n^2(n-1)^2}{4}, \quad \alpha, \beta, \gamma \in \mathbb{R}. \end{aligned}$$

After finding $a(n)$, $b(n)$ and $c(n)$. The characteristic equation is

$$Q(n, x_n) = \left(\gamma - \beta\frac{n(n-1)}{2} + \alpha\frac{n^2(n-1)^2}{4}\right)x_n^2 + \left(\beta - \alpha n(n-1)\right)x_n + \alpha.$$

It remains to find the genaral solution $\{x_n\}$, we use the following steps:

1. Determining the characteristic function $Q(n, x_n)$.

2. Finding the canonical coordinate y_n. To simplify the calculations we assume that $\hat{Q}(n, y_n) = 1$, then

$$y(n, x_n) = \int \frac{dx_n}{Q(n, x_n)}.$$

3. From step 2, we have y as a function of n and x_n, we write a difference equation of y_n, and solve a first order difference equation.

4. Write the solution in step 3 in terms of x_n, and note that this happens only if we can invert the map x_n to y_n. This condition is called a compatible canonical coordinate.

In the following example, we illustrate the above steps.

Example 3.17. [6] Find the general solution $\{x_n\}$, for

$$x_{n+1} = \frac{x_n}{1+nx_n}, \quad n \geq 1.$$

Solution: Now we use the canonical coordinate which is injective after we find the characteristic function. Then, we find the genaral solution as follow: from pervious example, we find the characteristic $Q(n, x_n)$ which is

$$Q(n,x_n) = \left(\gamma - \beta\frac{n(n-1)}{2} + \alpha\frac{n^2(n-1)^2}{4}\right)x_n^2 + \left(\beta - \alpha n(n-1)\right)x_n + \alpha.$$

To simplify calculations, we assume $\alpha = \beta = 0$ and $\gamma = 1$. Therefore, $Q(n, x_n) = x_n^2$. The canonical coordinate is

$$y(n, x_n) = \int \frac{dx_n}{x_n^2} = \frac{-1}{x_n} + c, c \in \mathbb{R},$$

which is invertible, that is we can write x_n in terms of y_n. Consider the difference equation

$$y_{n+1} - y_n = \frac{-1}{x_{n+1}} - \frac{-1}{x_n}.$$

If we substitute $x_{n+1} = \frac{x_n}{1+nx_n}$, we get a first order difference equation in y_n,

$$y_{n+1} - y_n = -n,$$

which has general solution:

$$y_n = m - \frac{n(n-1)}{2}; \quad m \in \mathbb{R}.$$

Since $y_n = \frac{-1}{x_n}$, so the general solution of the original difference equation is:

$$x_n = \frac{2}{-2m + n(n-1)}; \quad n \geq 1, m \in \mathbb{R}.$$

Remark 3.18. For a differnce equation, we have more than one characteristic function that depends on constants. Every time we choose the constants that simplify our calculations.

3.1.2 Lie Symmetries for $2^{nd} O\Delta E$

In this section, we need to solve a given second order difference equation by Lie symmetries. The idea is, to find the linearized symmetry condition (LSC) for second order difference equations, as we did for first order difference equations.

Now, consider the difference equation

$$x_{n+2} = f(n, x_n, x_{n+1}); \quad n \in \mathbb{Z}, \tag{3.22}$$

we assume that $\frac{\partial f}{\partial x_{n+1}} \neq 0$, (this condition ensures that the equation is truely second order), the symmetry condition is

$$\hat{x}_{n+2} = f(\hat{n}, \hat{x}_n, \hat{x}_{n+1}), \quad \text{when Eq.(3.22) holds.} \tag{3.23}$$

As we did for first order equations, from symmetry condition Eq.(3.23), we have

$$f(\hat{n}, \hat{x}_n, \hat{x}_{n+1}) = f(n, x_n + \alpha Q(n, x_n), x_{n+1} + \alpha Q(n+1, x_{n+1})). \tag{3.24}$$

Need to find Taylor series for the right hand side about $\alpha = 0$, we get

$$\begin{aligned} f(\hat{n}, \hat{x}_n, \hat{x}_{n+1}) &= f(n, x_n, x_{n+1}) + \alpha\left(\frac{\partial f}{\partial \hat{x}_{n+1}} \frac{\partial \hat{x}_{n+1}}{\partial \alpha} |_{\alpha=0} + \frac{\partial f}{\partial \hat{x}_n} \frac{\partial \hat{x}_n}{\partial \alpha} |_{\alpha=0}\right) \\ &\quad + O(\alpha^2) \\ &= f(n, x_n, x_{n+1}) + \alpha\left(\frac{\partial f}{\partial x_{n+1}} Q(n+1, x_{n+1}) + \frac{\partial f}{\partial x_n} Q(n, x_n)\right) \\ &\quad + O(\alpha^2), \end{aligned} \tag{3.25}$$

also we have

$$f(\hat{n}, \hat{x}_n, \hat{x}_{n+1}) = \hat{f}(n, x_n, x_{n+1}) = \hat{x}_{n+2} = f(n, x_n, x_{n+1}) + \alpha Q(n+2, x_{n+2}) + O(\alpha^2). \tag{3.26}$$

By comparing Eq.(3.25) and Eq.(3.26),we get the linearized symmetry condition (LSC) for second order difference equation

$$Q(n+2, x_{n+2}) = \frac{\partial f}{\partial x_{n+1}} Q(n+1, x_{n+1}) + \frac{\partial f}{\partial x_n} Q(n, x_n)$$

We need to simplify this formula, since it is a functional equation which is hard to solve. The more important concept that plays a big role in simplifing (LSC) for second order and higher orders is the infinitesimal generator.

Definition 3.19. [7] *The infinitesimal generator I for a differnce equation of order p is*

$$I = \sum_{k=0}^{p-1} (S^k Q(n, x_n)) \frac{\partial}{\partial x_{n+k}},$$

where S^k is the forward shift operator such that $S^k x_n = x_{n+k}$.

Therefore, by the above definition the Linearized symmetry condition for second order difference equations becomes

$$\begin{aligned}
Q(n+2,x_{n+2}) &= \frac{\partial f}{\partial x_n}Q(n,x_n)+\frac{\partial f}{\partial x_{n+1}}Q(n+1,x_{n+1}) \\
S^2Q(n,x_n) &= S^0Q(n,x_n)\frac{\partial f}{\partial x_n}+SQ(n+1,x_{n+1})\frac{\partial f}{\partial x_{n+1}} \\
S^2Q(n,x_n) &= \sum_{k=0}^{1}(S^kQ(n,x_n))\frac{\partial f}{\partial x_{n+k}} \\
S^2Q &= If \qquad (3.27)
\end{aligned}$$

Eq.(3.27) is a linear functional equation for the characteristics $Q(n, x_n)$. But functional equations are generally hard to solve. Luckily, Lie symmetries are diffeomorphisms, that is, $Q(n, x_n)$ is a smooth function, this implies that the linearized symmetry condition can be solved by the method of differential elimination. That is, we transform Eq.(3.27) from a functional equation into a differential equation. We consider the difference equations that satisfies the conditions $\frac{\partial f}{\partial x_{n+1}} \neq 0$ and $\frac{\partial f}{\partial x_n} \neq 0$. We follow two steps.

Firstly, by elliminating $Q(n + 2, x_{n+2})$ and $Q(n + 1, x_{n+1})$, we can form an ordinary differential equation of $Q(n, x_n)$.
To achieve this objective we differentiate the linearized symmetry condition with respect to x_n keeping f fixed and we consider x_{n+1} to be a function of n, x_n and f. Therefore, we apply the differential operator (L)

$$L = \frac{\partial}{\partial x_n} + \frac{\partial x_{n+1}}{\partial x_n}\frac{\partial}{\partial x_{n+1}},$$

but

$$\frac{\partial x_{n+1}}{\partial x_n} = -\frac{\partial f/\partial x_n}{\partial f/\partial x_{n+1}}.$$

The first term of the functional equation Eq.(3.27) is elliminated by this differential operator, since we differentiate with respect to x_n keeping f fixed, so we obtain

$$\frac{\partial}{\partial x_n}\Bigg(Q(n+2,f)\Bigg)=0,$$

$$\frac{\partial}{\partial x_n}\Bigg(\frac{\partial f}{\partial x_n}Q(n,x_n)\Bigg)=\frac{\partial f}{\partial x_n}Q'(n,x_n)+\frac{\partial^2 f}{\partial x_n^2}Q(n,x_n),$$

$$\frac{\partial}{\partial x_n}\Bigg(\frac{\partial f}{\partial x_{n+1}}Q(n+1,x_{n+1})\Bigg)=\frac{\partial^2 f}{\partial x_n\partial x_{n+1}}Q(n+1,x_{n+1}),$$

and

$$\frac{\partial}{\partial x_{n+1}}\Bigg(Q(n+2,f)\Bigg)=0,$$

$$\frac{\partial}{\partial x_{n+1}}\Bigg(\frac{\partial f}{\partial x_n}Q(n,x_n)\Bigg)=\frac{\partial^2 f}{\partial x_{n+1}\partial x_n}Q(n,x_n),$$

$$\frac{\partial}{\partial x_{n+1}}\Bigg(\frac{\partial f}{\partial x_{n+1}}Q(n+1,x_{n+1})\Bigg)=\frac{\partial f}{\partial x_{n+1}}Q'(n+1,x_{n+1})+\frac{\partial^2 f}{\partial x_{n+1}^2}Q(n+1,x_{n+1}).$$

This implies that

$$\Bigg(-\frac{\partial f}{\partial x_n}Q'(n,x_n)-\frac{\partial^2 f}{\partial x_n^2}Q(n,x_n)-\frac{\partial^2 f}{\partial x_n\partial x_{n+1}}Q(n+1,x_{n+1})\Bigg)+$$
$$\Bigg(\frac{\partial x_{n+1}}{\partial x_n}\Bigg)\Bigg(-\frac{\partial^2 f}{\partial x_{n+1}\partial x_n}Q(n,x_n)-\frac{\partial f}{\partial x_{n+1}}Q'(n+1,x_{n+1})-\frac{\partial^2 f}{\partial x_{n+1}^2}Q(n+1,x_{n+1})\Bigg)=0.$$

Secondly, we elliminate $Q'(n+1,x_{n+1})$.

By differentiating the equation obtained in the previous step with respect to x_n keeping x_{n+1} fixed. We may have to differentiate once more with respect to x_n keeping x_{n+1} fixed. After that, we obtain an ordinary differential equation, which can be split by gathering together all terms with the same dependence upon x_{n+1} and we solve it if possible, and obtain $Q(n,x_n)$. To find the coefficients of the terms of $Q(n,x_n)$, we plug it in the equations that we obtained in previous steps which can be split into a system of linear difference equations by collecting all terms with the same dependence on x_n and x_{n+1}.

Therefore, we will have a characteristic function $Q(n, x_n)$ for second order difference equations. The following example illustrates the above steps.

Example 3.20. [12] Find the characteristic functions of the following second order difference equation

$$x_{n+2} = \frac{a x_n x_{n+1}}{x_n + x_{n+1}}; \quad a \in \mathbb{R}^*. \tag{3.28}$$

Solution: Note that

$$f(n, x_n, x_{n+1}) = \frac{a x_n x_{n+1}}{x_n + x_{n+1}}.$$

The *LSC* of Eq.(3.28) is

$$Q(n+2, x_{n+2}) - \frac{\partial f}{\partial x_n} Q(n, x_n) - \frac{\partial f}{\partial x_{n+1}} Q(n+1, x_{n+1}) = 0,$$

Now need to find $\frac{\partial f}{\partial x_n}$ and $\frac{\partial f}{\partial x_{n+1}}$.

So,

$$\frac{\partial f}{\partial x_n} = \frac{a x_{n+1}^2}{(x_n + x_{n+1})^2} = \frac{f^2}{a x_n^2},$$

and

$$\frac{\partial f}{\partial x_{n+1}} = \frac{a x_n^2}{(x_n + x_{n+1})^2} = \frac{f^2}{a x_{n+1}^2},$$

so the *LSC* is

$$Q(n+2, x_{n+2}) - \frac{f^2}{a x_n^2} Q(n, x_n) - \frac{f^2}{a x_{n+1}^2} Q(n+1, x_{n+1}) = 0. \tag{3.29}$$

Second step is, to apply an appropriate diffferentail operator L, to reduce the number of unknown functions $Q(n+2, x_{n+2})$ and $Q(n+1, x_{n+1})$, and to transform this functional equation to differential equation. Note that $\frac{\partial x_{n+1}}{\partial x_n} = -\frac{x_{n+1}^2}{x_n^2}$,

$$\begin{aligned} L &= \frac{\partial}{\partial x_n} + \frac{\partial x_{n+1}}{\partial x_n}\frac{\partial}{\partial x_{n+1}}, \\ &= \frac{\partial}{\partial x_n} - \frac{x_{n+1}^2}{x_n^2}\frac{\partial}{\partial x_{n+1}}. \end{aligned}$$

Now, apply L to Eq.(3.28) to get

$$\left(\frac{\partial}{\partial x_n} - \frac{x_{n+1}^2}{x_n^2}\frac{\partial}{\partial x_{n+1}}\right)\left(Q(n+2, x_{n+2}) - \frac{f^2}{ax_n^2}Q(n, x_n) - \frac{f^2}{ax_{n+1}^2}Q(n+1, x_{n+1})\right) = 0, \tag{3.30}$$

but we have,

$$\begin{aligned} &\frac{\partial}{\partial x_n}(Q(n+2, x_{n+2})) = 0, \\ &\frac{\partial}{\partial x_n}\left(\frac{f^2}{ax_n^2}Q(n, x_n)\right) = \frac{f^2}{ax_n^2}Q'(n, x_n) - \frac{2f^2}{ax_n^3}Q(n, x_n), \\ &\frac{\partial}{\partial x_n}\left(\frac{f^2}{ax_{n+1}^2}Q(n+1, x_{n+1})\right) = 0, \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial x_{n+1}}(Q(n+2, x_{n+2})) = 0, \\ &\frac{\partial}{\partial x_{n+1}}\left(\frac{f^2}{ax_n^2}Q(n, x_n)\right) = 0, \\ &\frac{\partial}{\partial x_{n+1}}\left(\frac{f^2}{ax_{n+1}^2}Q(n+1, x_{n+1})\right) = \frac{f^2}{ax_{n+1}^2}Q'(n+1, x_{n+1}) + \frac{-2f^2}{ax_{n+1}^3}Q(n+1, x_{n+1}). \end{aligned}$$

From the above calculations Eq.(3.30) can be written as

$$\frac{-f^2}{ax_n^2}Q'(n, x_n) + \frac{2f^2}{ax_n^3}Q(n, x_n) - \frac{x_{n+1}^2}{x_n^2}\left(\frac{-f^2}{ax_{n+1}^2}Q'(n+1, x_{n+1}) + \frac{2f^2}{ax_{n+1}^3}Q(n+1, x_{n+1})\right) = 0,$$

multiplying the last equation by $\frac{-ax_n^2}{f^2}$, we get

$$Q'(n, x_n) - \frac{2}{x_n}Q(n, x_n) - Q'(n+1, x_{n+1}) + \frac{2}{x_{n+1}}Q(n+1, x_{n+1}) = 0. \tag{3.31}$$

Now, we differentiate Eq.(3.31) with respect to x_n keeping x_{n+1} fixed, we obtain

$$\frac{\partial}{\partial x_n}(Q'(n,x_n) - \frac{2}{x_n}Q(n,x_n) - Q'(n+1,x_{n+1}) + \frac{2}{x_{n+1}}Q(n+1,x_{n+1})) = 0,$$

but

$$\begin{aligned}
&\frac{\partial}{\partial x_n}(Q'(n,x_n)) = Q''(n,x_n),\\
&\frac{\partial}{\partial x_n}(\frac{2}{x_n}Q(n,x_n)) = \frac{2}{x_n}Q'(n,x_n) + \frac{-2}{x_n^2}Q(n,x_n),\\
&\frac{\partial}{\partial x_n}(Q'(n+1,x_{n+1})) = 0,\\
&\frac{\partial}{\partial x_n}(\frac{2}{x_{n+1}}Q(n+1,x_{n+1})) = 0,
\end{aligned}$$

so

$$Q''(n,x_n) - \frac{2}{x_n}Q'(n,x_n) + \frac{2}{x_n^2}Q(n,x_n) = 0,$$

multiply this equation by x_n^2, we get

$$x_n^2 Q''(n,x_n) - 2x_n Q'(n,x_n) + 2Q(n,x_n) = 0,$$

Note that, we got a differential equation in $Q(n,x_n)$, which is an Euler differential equation whose solution is given by

$$Q(n,x_n) = \lambda(n)x_n^2 + \eta(n)x_n,$$

for some functions λ and η of n. Note that

$$Q'(n,x_n) = 2\lambda(n)x_n + \eta(n),$$

substitue Q and Q' into Eq.(3.31) we get

$$\begin{aligned}
2\lambda(n)x_n + \eta(n) - \frac{2}{x_n}(\lambda(n)x_n^2 + \eta(n)x_n) - 2\lambda(n+1)x_{n+1} - \eta(n+1)+&\\
\frac{2}{x_{n+1}}(\lambda(n+1)x_{n+1}^2 + \eta(n+1)x_{n+1}) = 0,&
\end{aligned}$$

simplifying, we get

$$2\lambda(n)x_n+\eta(n)-2\lambda(n)x_n-2\eta(n)-2\lambda(n+1)x_{n+1}-\eta(n+1)+2\lambda(n+1)x_{n+1}+2\eta(n+1)=0.$$

Hence, the only terms that we have are

$$\eta(n+1)=\eta(n),$$

which is a first order linear homogeneuous difference equation whose solution is

$$\eta(n)=c, \quad c\in\mathbb{R}.$$

Now, Q becomes $Q(n,x_n)=\lambda(n)x_n^2+cx_n$, substitute $\eta(n)=c$ in the LSC to obtain

$$\lambda(n+2)x_{n+2}^2+cx_{n+2}-\frac{f^2}{ax_n^2}(\lambda(n)x_n^2+cx_n)-\frac{f^2}{ax_{n+1}^2}(\lambda(n+1)x_{n+1}^2+cx_{n+1})=0, \tag{3.32}$$

substitute $x_{n+2}=f$, to get

$$\left(\lambda(n+2)f^2-\frac{f^2}{a}\lambda(n)-\frac{f^2}{a}\lambda(n+1)\right)+\left(cf-cf\frac{f}{ax_n}-cf\frac{f}{ax_{n+1}}\right)=0, \tag{3.33}$$

note that the second parentheses is 0 since

$$cf-cf\frac{f}{ax_n}-cf\frac{f}{ax_{n+1}}=cf-cf\frac{x_{n+1}}{x_n+x_{n+1}}-cf\frac{x_n}{x_n+x_{n+1}}=$$
$$cf\left(1-\frac{x_n}{x_n+x_{n+1}}-\frac{x_{n+1}}{x_n+x_{n+1}}\right)=cf\left(\frac{0}{x_n+x_{n+1}}\right)=0.$$

Therefore, Eq.(3.33) implies

$$\lambda(n+2)f^2-\frac{f^2}{a}\lambda(n)-\frac{f^2}{a}\lambda(n+1)=0,$$

hence,

$$\lambda(n+2)-\frac{1}{a}\lambda(n)-\frac{1}{a}\lambda(n+1)=0,$$

which is a second order difference equation, to solve it, we find it's characteristic equation

$$\begin{aligned} r^{n+2} - \frac{1}{a} r^{n+1} - \frac{1}{a} r^{n} &= 0, \\ a r^2 - r - 1 &= 0. \end{aligned} \tag{3.34}$$

The roots of Eq.(3.34) are

$$r_{1,2} = \frac{1 \pm \sqrt{1+4a}}{2a}.$$

We have two cases depending on the value of a:

1. If $a = \frac{-1}{4}$, then $r = -2$ (two roots real and repeated). It follows

$$\lambda(n) = c_1(-2)^n + c_2 n(-2)^n, \quad c_1, c_2 \in \mathbb{R},$$

so the characteristic function is

$$Q(n, x_n) = (c_1(-2)^n + c_2 n(-2)^n) x_n^2 + c x_n,$$

2. $If a \neq \frac{-1}{4}$, then

$$\lambda(n) = c_1 \left(\frac{1+\sqrt{1+4a}}{2a} \right)^n + c_2 \left(\frac{1-\sqrt{1+4a}}{2a} \right)^n,$$

so the characteristic function is

$$Q(n, x_n) = \left(c_1 \left(\frac{1+\sqrt{1+4a}}{2a} \right)^n + c_2 \left(\frac{1-\sqrt{1+4a}}{2a} \right)^n \right) x_n^2 + c x_n,$$

where c, c_1 and $c_2 \in \mathbb{R}$.

Now, we use symmetries to reduce the order of difference equations. We find a compatible canonical coordinate, which reduces the order by one. If the reduced equation can be solved, then the original equation can be solved by one more integration or summation.

Definition 3.21. [10] *A function v_n is invariant under the Lie group of transformations T_h if $Iv_n = 0$, where I is the infinitesimal generator, such that $I = \sum_{k=0}^{p-1} S^k Q(n, x_n) \frac{\partial}{\partial x_{n+k}}$.*

Suppose that the characteristic $Q(n, x_n)$ for the second order difference equation

$$x_{n+2} = f(n, x_n, x_{n+1}),$$

is known, then the invariant v_n can be found by solving the partial differential equation

$$Iv_n = Q(n, x_n)\frac{\partial v_n}{\partial x_n} + Q(n+1, x_{n+1})\frac{\partial v_n}{\partial x_{n+1}} = 0,$$

which is a quasi linear partial differential equation that can be solved using the method of characteristics, set

$$\frac{dx_n}{Q(n, x_n)} = \frac{dx_{n+1}}{SQ(n, x_n)} = \frac{dv_n}{0}. \tag{3.35}$$

If the invariant function $v_{n+1}(n, x_n, x_{n+1})$ can be written as a function of n and v_n only, then v_n can reduce the order of the difference equation by *one* to obtain

$$x_{n+1} = g(n, x_n, v_n),$$

for some function g. This equation is first order difference equation.
Finally, as we mentioned in the previous section, to solve the first order equation, we need to obtain a canonical coordinate s_n.

Example 3.22. [6] Find the general solution for the previous example, with $a = 2$,

$$x_{n+2} = \frac{2x_n x_{n+1}}{x_n + x_{n+1}}.$$

If $a = 2$, then $Q(n, x_n) = \left(c_1 + c_2(\frac{-1}{2})^n\right) x_n^2 + cx_n$. Now we will choose c, c_1, and c_2 such that we simplify our calculations. Therefore, if $c_1 = 1$

and $c = c_2 = 0$. Then $Q(n, x_n) = x_n^2$. Next step is to find the canonical coordinate s_n, also for second order equations we assume that $\hat{Q}(n, s_n) = 1$, implies

$$s_n = \int \frac{dx_n}{Q(n, x_n)} = \int \frac{dx_n}{x_n^2} = \frac{-1}{x_n}.$$

By Eq.(3.35), the invarient v_n is given by

$$\frac{dx_n}{x_n^2} = \frac{dx_{n+1}}{x_{n+1}^2} = \frac{dv_n}{0}.$$

Taking the second and first invarients, we have

$$\int \frac{dx_{n+1}}{x_{n+1}^2} = \int \frac{dx_n}{x_n^2},$$

$$\frac{-1}{x_{n+1}} = \frac{-1}{x_n} + c_1, \quad \text{which implies} \quad c_1 = \frac{-1}{x_{n+1}} - \frac{-1}{x_n}, \quad c_1 \in \mathbb{R}.$$

Taking the third and first invarients, we have

$$\frac{dv_n}{0} = \frac{dx_n}{x_n^2},$$

by reciprocal multiplication, we get $dv_n = 0$. Therefore,

$$v_n = c_2, \quad c_2 \in \mathbb{R},$$

such that $c_2 = g(c_1)$ where g is an arbitrary function. We will choose it to be the identity such that $g(c_1) = c_1$ and hence $c_2 = c_1$.

Therefore,

$$v_n = c_2 = \frac{1}{x_n} - \frac{1}{x_{n+1}}.$$

Applying the shift operator to v_n, we get

$$\begin{aligned} v_{n+1} &= \frac{1}{x_{n+1}} - \frac{1}{x_{n+2}} \\ &= \frac{1}{x_{n+1}} - \frac{x_{n+1} + x_n}{2x_n x_{n+1}} \\ &= \frac{1}{2x_{n+1}} - \frac{1}{2x_n} \\ &= -\frac{v_n}{2}. \end{aligned}$$

So, we have a first order linear homogeneuous difference equation:

$$v_{n+1} + \frac{v_n}{2} = 0,$$

whose solution is given by

$$v_n = c_3 \left(\frac{-1}{2}\right)^n, \quad where\ c_3 \in \mathbb{R}.$$

It follows that

$$s_{n+1} - s_n = \frac{-1}{x_{n+1}} - \frac{-1}{x_n} = v_n = c_3 \left(\frac{-1}{2}\right)^n.$$

This equation is a first order linear non-homogeneuous difference equation whose solution is given by

$$\begin{aligned} s_n = s_0 + \sum_{k=0}^{n-1} c_3 \left(\frac{-1}{2}\right)^k &= s_0 + c_3 \frac{(1-(\frac{-1}{2})^n)}{1-\frac{-1}{2}} \\ &= s_0 + c_3 \frac{2(1-(\frac{-1}{2})^n)}{3}, \end{aligned}$$

but $s_n = \frac{-1}{x_n}$, so

$$\begin{aligned} x_n &= \frac{-1}{s_0 + c_3 \frac{2(1-(\frac{-1}{2})^n)}{3}} \\ &= \frac{-1}{\frac{-1}{x_0} + c_3 \frac{2(1-(\frac{-1}{2})^n)}{3}} \\ &= \frac{1}{\frac{1}{x_0} - \frac{2}{3} c_3 (1-(\frac{-1}{2})^n)} \\ &= \frac{1}{(\frac{1}{x_0} - \frac{2c_3}{3}) + \frac{2c_3}{3}(-2)^{-n}} \\ &= \frac{1}{\hat{c}_1 + \hat{c}_2 (-2)^{-n}}, \end{aligned}$$

where $\hat{c}_1$ and $\hat{c}_2 \in \mathbb{R}$, and they are not both *zero*.

In short, the general technique for obtaining Lie point symmetry of any difference equation of order $k \geq 2$ is

1. Write the LSC.
2. Apply appropriate differential operators to reduce the number of unknown functions.
3. Having reached a differential equation, back-substitute and solve the resulting linear difference equation.
4. Iterate, if necessary.

4 HOMOGENEOUS DIFFERENCE EQUATIONS

In this chapter, we will talk about reduction of order method for homogeneous difference equations of degree one denoted by ($HD1$), and we consider some examples. Then, we provide Euler's theorem for homogeneuous functions that we use in explaining the relation between reduction of order for $HD1$ and Lie symmetry method.

4.1 Introduction

In this section, we present what do we mean by a homogeneuous difference equation of order k in both cases multiplicative and additive.

Definition 4.1. [4] *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called **multiplicative homogeneous of degree** k ;for short MHk; if*

$$f(tx_1, \cdots, tx_n) = t^k f(x_1, \cdots, x_n)$$

for all $t \in \mathbb{R}^+$.

Definition 4.2. [4] *A function* $f : \mathbb{R}^n \to \mathbb{R}$ *is called* ***additive homogeneous of degree*** k *;for short* AHk*; if*

$$f(t + x_1, \cdots, t + x_n) = t^k + f(x_1, \cdots, x_n)$$

for all $t \in \mathbb{R}^+$.

A special case if $k = 1$, then the function f is called homogeneuous of degree one ($HD1$). As a result, we say ($MH1$) and ($AH1$) in multiplicative and additive cases, respectively.

4.2 Order Reduction Theorem For HD1

In this section, we present a theorem that plays an important role in reducing the order of difference equations in both cases multiplicative and additive. H. Sedaghat in [5] shows that every difference equation of order $k + 1$

$$x_{n+1} = f_n(x_n, x_{n-1}, \cdots, x_{n-k}), \tag{4.1}$$

with each mapping f_n being homogeneous of degree one with positive initial conditions is equivalent to a system that consists of an equation of order k and a linear equation of order one.

Definition 4.3. [5] The difference equation Eq.(4.1) is said to be $MH1$ if f_n is $MH1, \forall\ n = 0, 1, \cdots$.

Definition 4.4. [5] The difference equation Eq.(4.1) is said to be $AH1$ if f_n is $AH1$, $\forall\ n = 0, 1, \cdots$.

We have the following theorem.

Theorem 4.5. [5] *Let G be a nontrivial group. And let*

$$x_{n+1} = f_n(x_n, x_{n-1}, \cdots, x_{n-k}), \tag{4.2}$$

be an equation of order $k+1$. Then,

1. *If f_n is $MH1$ relative to G for all $n \geq 1$, then Eq.(4.2) is equivalent to the following system of equations*

$$\begin{aligned} r_{n+1} &= f_n(1, r_n^{-1}, (r_{n-1}r_n)^{-1}, \cdots, (r_{n-k+1}\cdots r_{n-1}r_n)^{-1}), & (4.3)\\ s_{n+1} &= s_n r_{n+1}. & (4.4) \end{aligned}$$

2. *If f_n is $AH1$ relative to G for all $n \geq 1$, then Eq.(4.2) is equivalent to the following system of equations*

$$\begin{aligned} r_{n+1} &= f_n(0, -r_n, -r_{n-1} - r_n, \cdots, -r_{n-k+1} - \cdots - r_n), & (4.5)\\ s_{n+1} &= s_n + r_{n+1}. & (4.6) \end{aligned}$$

Where the first equation in each system is of order k and the second is linear in s_n of order 1.

Proof. 1. suppose for each solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(4.2), we define

$$r_n = x_{n-1}^{-1}x_n, \forall n = -k+1, -k+2, \cdots.$$

Then,

$$\begin{aligned} r_{n+1} &= x_n^{-1}x_{n+1} \\ &= x_n^{-1}f_n(x_n, x_{n-1}, x_{n-2}, \cdots, x_{n-k}) \\ &= f_n(x_n^{-1}x_n, x_n^{-1}x_{n-1}, x_n^{-1}x_{n-2}, \cdots, x_n^{-1}x_{n-k}) \end{aligned}$$

$$= f_n(1, x_n^{-1}x_{n-1}, (x_n^{-1}x_{n-1})(x_{n-1}^{-1}x_{n-2}), \cdots, (x_n^{-1}x_{n-1})(x_{n-1}^{-1}x_{n-2}) \cdots (x_{n-k+1}^{-1}x_{n-k}))$$

$$= f_n(1, r_n^{-1}, (r_{n-1}r_n)^{-1}, \cdots, (r_{n-k+1}\cdots r_{n-1}r_n)^{-1}).$$

It follows that $\{r_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.3). And since $x_{n+1} = x_n r_{n+1}$, let $s_n = x_n$ for $n = -k+1, -k+2, \cdots$, we have

$$s_{n+1} = s_n r_{n+1}.$$

It follows that $\{s_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.4), so that $\{(r_n, s_n)\}_{n=-k+1}^{\infty}$ is a solution of the system.

Conversely, suppose $\{(r_n, s_n)\}_{n=-k+1}^{\infty}$ be a solution of the system. Then $\{r_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.3) and $\{s_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.4). Choose $x_{-k} \in G$ and set $x_n = s_n$ for $n = -k+1, -k+2, \cdots$. Then,

$$\begin{aligned}
x_{n+1} &= s_{n+1} \\
&= s_n r_{n+1} \\
&= x_n f_n(1, r_n^{-1}, (r_{n-1}r_n)^{-1}, \cdots, (r_{n-k+1}\cdots r_{n-1}r_n)^{-1}) \\
&= f_n(x_n, x_n(x_{n-1}^{-1}x_n)^{-1}, x_n(x_{n-2}^{-1}x_n)^{-1}, \cdots, x_n(x_{n-k}^{-1}x_n)^{-1}) \\
&= f_n(x_n, x_{n-1}, \cdots, x_{n-k}).
\end{aligned}$$

It follows that $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(4.2).

2. suppose for each solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(4.2), we define

$$r_n = x_n - x_{n-1}, \forall n = -k+1, -k+2, \cdots.$$

Then,

$$\begin{aligned}
r_{n+1} &= x_{n+1} - x_n \\
&= -x_n + f_n(x_n, x_{n-1}, x_{n-2}, \cdots, x_{n-k}) \\
&= f_n(-x_n + x_n, -x_n + x_{n-1}, -x_n + x_{n-2}, \cdots, -x_n + x_{n-k})
\end{aligned}$$

$$= f_n(0, -x_n + x_{n-1}, (-x_n + x_{n-1}) + (-x_{n-1} + x_{n-2}), \cdots, \\ (-x_n + x_{n-1}) + (-x_{n-1} + x_{n-2}) + \cdots + (-x_{n-k+1} + x_{n-k}))$$

$$= \quad f_n(0, -r_n, -r_n - r_{n-1}, \cdots, -r_n - r_{n-1} - \cdots - r_{n-k+1}).$$

It follows that $\{r_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.5). And since $x_{n+1} = x_n + r_{n+1}$, let $s_n = x_n$ for $n = -k+1, -k+2, \cdots$, we have

$$s_{n+1} = s_n + r_{n+1}.$$

It follows that $\{s_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.6). so that $\{(r_n, s_n)\}_{n=-k+1}^{\infty}$ is a solution of the system.

Conversely, suppose $\{(r_n, s_n)\}_{n=-k+1}^{\infty}$ be a solution of the system. Then $\{r_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.5) and $\{s_n\}_{n=-k+1}^{\infty}$ is a solution of Eq.(4.6). Choose $x_{-k} \in G$ and set $x_n = s_n$ for $n = -k+1, -k+2, \cdots$. Then,

$$\begin{aligned} x_{n+1} &= s_{n+1} \\ &= s_n + r_{n+1} \\ &= x_n + f_n(0, -r_n, -r_n - r_{n-1}, \cdots, -r_n - r_{n-1} - \cdots - r_{n-k+1}) \\ &= f_n(x_n + 0, x_n - r_n, x_n - r_n - r_{n-1}, \cdots, x_n - r_n - r_{n-1} - \cdots - r_{n-k+1}) \\ &= f_n(x_n, x_{n-1}, \cdots, x_{n-k}). \end{aligned}$$

It follows that $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(4.2).

□

Remark 4.6. [5]

1. We can construct the first equation in each of the above systems in the previous theorem quickly by substitution as follows

If f is $MH1$, then

$$1 \longrightarrow x_n, (r_{n-i+1} \cdots r_{n-1} r_n)^{-1} \longrightarrow x_{n-i}$$

for $i = 1, 2, \cdots, k$.

And if f is $AH1$, then

$$0 \longrightarrow x_n, -r_{n-i+1} - \cdots - r_{n-1} - r_n \longrightarrow x_{n-i}$$

for $i = 1, 2, \cdots, k$.

2. The system in the above theorem in each case can be solved explicitly in terms of a solution $\{r_n\}_{n=-k+1}^{\infty}$ as follows

   If f is $MH1$, then

$$s_n = s_0 r_1 r_2 \cdots r_n, n = 1, 2, 3, \cdots \quad ,$$

   and since $x_n = s_n$ for $n = -k+1, -k+2, \cdots$. Therefore, the solution of Eq.(4.2) is

$$x_n = x_0 \prod_{i=1}^{n} r_i.$$

   And if f is $AH1$, then

$$s_n = s_0 + r_1 + r_2 + \cdots + r_n, n = 1, 2, 3, \cdots \quad .$$

   It follows that the solution of Eq.(4.2) is

$$x_n = x_0 + \sum_{i=1}^{n} r_i.$$

   Thus for $HD1$ functions in both cases $MH1$ and $AH1$, the above theorem essentially reduces the study of Eq.(4.2) with order $k+1$ to that of the first equation of the above system which is of order k.

Here, we will discuss equations of order $k \geq 2$, to illustrate the above theorem.

Example 4.7. [4] Consider the non-autonomous **second order** rational difference equation

$$x_{n+1} = \frac{a_n x_n^2 + b_n x_n x_{n-1}}{c_n x_{n-1}}, \tag{4.7}$$

where $a_n, b_n \geq 0$ with $a_n + b_n > 0$ and $c_n > 0$ for all n. The previous equation can be written as

$$x_{n+1} = \frac{\alpha_n x_n^2 + \beta_n x_n x_{n-1}}{x_{n-1}}, \tag{4.8}$$

where $\alpha_n = \frac{a_n}{c_n}$, $\beta_n = \frac{b_n}{c_n}$.

f_n is $MH1$ under multiplication for all $n \geq 1$, since $\forall t > 0$,

$$\begin{aligned} f_n(tx_n, tx_{n-1}) &= \frac{\alpha_n (tx_n)^2 + \beta_n (tx_n)(tx_{n-1})}{tx_{n-1}} \\ &= \frac{t^2(\alpha_n x_n^2 + \beta_n x_n x_{n-1})}{tx_{n-1}} \\ &= t f_n(x_n, x_{n-1}). \end{aligned}$$

By the above theorem Eq.(4.7) is equivalent to the following system

$$\begin{aligned} r_{n+1} &= f_n(1, r_n^{-1}), \\ s_{n+1} &= s_n r_{n+1}. \end{aligned}$$

Now, solving the first equation of the system

$$\begin{aligned} r_{n+1} &= f_n(1, r_n^{-1}) \\ &= f_n(1, \frac{1}{r_n}) \\ &= \alpha_n r_n + \beta_n. \end{aligned}$$

Therefore, r_{n+1} is a first order linear non-homogeneous difference equation, whose solution is

$$r_n = \prod_{i=0}^{n-1} \alpha_i r_0 + \sum_{i=0}^{n-1} (\prod_{j=i}^{n-1} \alpha_j) \beta_i \tag{4.9}$$

where r_0 is an initial value for r_n.

Thus, the solution of Eq.(4.8) can be written as follows

$$x_n = x_0 \prod_{k=1}^{n} r_k \tag{4.10}$$

where r_k is given by Eq.(4.9).

Example 4.8. [4] Consider the non-autonomous **second order** difference equation

$$x_{n+1} = a_n + x_n + c_n(b_n + x_n - x_{n-1})^2 \tag{4.11}$$

where a_n, b_n and c_n are given sequences of real numbers. This example illustrates the additive case, f_n is $AH1$ for all $n \geq 1$ since $\forall t > 0$,

$$\begin{aligned} f_n(t + x_n, t + x_{n-1}) &= a_n + t + x_n + c_n(b_n + t + x_n - t - x_{n-1})^2 \\ &= t + f_n(x_n, x_{n-1}). \end{aligned}$$

Thus, Eq.(4.11) is equivalent to the following system according to theorem(4.5)

$$\begin{aligned} r_{n+1} &= f_n(0, -r_n) \\ s_{n+1} &= s_n r_{n+1} \end{aligned}$$

The first equation of the system becomes

$$r_{n+1} = a_n + c_n(b_n + r_n)^2 \tag{4.12}$$

which is a first order non-linear difference equation, after solving it, the solution of Eq. (4.11) is given as

$$x_n = x_0 + \sum_{i=1}^{n} r_i.$$

In particular, if $a_n \to 0$, $b_n \to 0$, and $c_n \to 1$. Then

$$r_{n+1} = r_n^2 \tag{4.13}$$

Eq.(4.13) can be solved recursively. Let r_0 be given, then

$$\begin{aligned}
&n=1, \quad r_1 = r_0^2 = r_0^{2^1},\\
&n=2, \quad r_2 = r_1^2 = r_0^4 = r_0^{2^2},\\
&n=3, \quad r_3 = r_2^2 = r_0^8 = r_0^{2^3},\\
&n=4, \quad r_4 = r_3^2 = r_0^{16} = r_0^{2^4},\\
&\vdots\\
&n=k, \quad r_k = r_{k-1}^2 = r_0^{2^k},
\end{aligned}$$

Therefore, the solution of Eq.(4.13) is

$$r_n = r_0^{2^n},$$

where $r_0 = x_0 - x_{n-1}$. And in this case, the solution x_n is

$$x_n = x_0 + \sum_{i=1}^{n} r_0^{2^i} = x_0 + \frac{r_0^2(1-r_0^{2^n})}{1-r_0^2} = x_0 + \frac{(x_0-x_{n-1})^2(1-(x_0-x_{n-1})^{2^n})}{1-(x_0-x_{n-1})^2}.$$

Example 4.9. [5] Consider the autonomous **third order** difference equation that is both $AH1$ and $MH1$

$$x_{n+1} = x_n + \frac{a(x_n - x_{n-1})^2}{x_{n-1} - x_{n-2}}, \tag{4.14}$$

where $a > 0$, and the initial conditions x_{-2}, x_{-1}, and x_0 are positive such that $x_0, x_{-2} \neq x_{-1}$.

We will solve Eq.(4.14) as an $AH1$. Therefore, it can be reduced to an equation of order two such that

$$r_{n+1} = f(0, -r_n, -r_n - r_{n-1}) = \frac{ar_n^2}{r_{n-1}}, \tag{4.15}$$

with $r_n = x_n - x_{n-1}$, note that $r_n \neq 0 \ \forall \ n \geq 1$.

The solution of Eq.(4.15) can be written with respect to r_n as

$$x_n = x_0 + \sum_{i=1}^{n} r_i, \tag{4.16}$$

Now, we need to find r_i and then substitute it in Eq.(4.16).

Note that Eq.(4.15) is $MH1$, hence we can reduce it to a first order difference equation such that if we consider

$$r_{n+1} = \frac{ar_n^2}{r_{n-1}} = g(r_n, r_{n-1}), \tag{4.17}$$

then it's reduced equation is

$$t_{n+1} = g(1, \frac{1}{t_n}) = at_n, \tag{4.18}$$

with $t_n = \frac{r_n}{r_{n-1}}$, and the solution of Eq.(4.17) using Eq.(4.18) can be written as

$$r_k = r_0 \prod_{i=1}^{k} t_i, \tag{4.19}$$

Eq.(4.18) is a first order linear homogeneous difference equation, its solution is given by

$$t_k = a^k t_0,$$

with $t_0 = \frac{r_0}{r_{-1}}$. And then Eq.(4.19) becomes

$$\begin{aligned} r_k &= r_0 \prod_{i=1}^{k} a^i t_0 \\ &= r_0 t_0^k (a^{1+2+\cdots+k}) \\ &= r_0 t_0^k a^{\frac{k(k+1)}{2}}. \end{aligned} \tag{4.20}$$

Substitute Eq.(4.19) into Eq.(4.16) to get

$$x_n = x_0 + r_0 \sum_{i=1}^{n} t_0^i a^{\frac{i(i+1)}{2}} = x_0 + r_0 \sum_{i=1}^{n} (t_0 a^{\frac{i+1}{2}})^i,$$

where $t_0 = \frac{x_0 - x_{-1}}{x_{-1} - x_{-2}}$.

Example 4.10. [5] Consider the autonomous rational difference equation of **order** $k+1$

$$x_{n+1} = x_n(\frac{ax_{n-k+1}}{x_{n-k}} + b), \tag{4.21}$$

where $a, b > 0$, and $a + b \neq 1$.

Eq.(4.21) is $MH1$, and it is reducible to an equation of order k.

Let $r_n = \frac{x_n}{x_{n-1}}$, then

$$\begin{aligned} \frac{x_{n+1}}{x_n} &= \frac{ax_{n-k+1}}{x_{n-k}} + b, \\ r_{n+1} &= ar_{n-k+1} + b. \end{aligned} \tag{4.22}$$

Eq.(4.22) can be written as:

$$r_{n+k} = ar_n + b, \tag{4.23}$$

Eq.(4.22) can be solved recursively. Let $r_0, r_1, \cdots, r_{k-1}$ be given, then

$$\begin{aligned} &n = 0, \quad r_k = ar_0 + b, \\ &n = 1, \quad r_{k+1} = ar_1 + b, \\ &\vdots \\ &n = k-1, \quad r_{2k-1} = ar_{k-1} + b, \end{aligned}$$

$$\begin{aligned} &n = k, \quad r_{2k} = ar_k + b = a^2 r_0 + ab + b, \\ &n = k+1, \quad r_{2k+1} = ar_{k+1} + b = a^2 r_1 + ab + b, \\ &\vdots \\ &n = 2k-1, \quad r_{3k-1} = ar_{2k-1} + b = a^2 r_{k-1} + ab + b, \end{aligned}$$

$$\begin{aligned} &n = 2k, \quad r_{3k} = ar_{2k} + b = a^3 r_0 + a^2 b + ab + b, \\ &n = 2k+1, \quad r_{3k+1} = ar_{2k+1} + b = a^3 r_1 + a^2 b + ab + b, \\ &\vdots \\ &n = 3k-1, \quad r_{4k-1} = ar_{3k-1} + b = a^3 r_{k-1} + a^2 b + ab + b. \end{aligned}$$

We note that

$$\begin{aligned} r_k &= ar_0 + b, \\ r_{2k} &= a^2 r_0 + ab + b, \\ r_{3k} &= a^3 r_0 + a^2 b + ab + b, \\ &\vdots \end{aligned}$$

we conclude that the formula of the solution with respect to r_0 is

$$r_n = r_{mk} = a^m r_0 + b \sum_{i=0}^{m-1} a^i, \quad m = 1, 2, 3, \cdots .$$

And

$$\begin{aligned} r_{k+1} &= ar_1 + b, \\ r_{2k+1} &= a^2 r_1 + ab + b, \\ r_{3k+1} &= a^3 r_1 + a^2 b + ab + b, \\ &\vdots \end{aligned}$$

We conclude that the formula of the solution with respect to r_1 is

$$r_n = r_{mk+1} = a^m r_1 + b \sum_{i=0}^{m-1} a^i, \quad m = 1, 2, 3, \cdots .$$

If we do the same notations and calculations as above, we will reach to the last r_{k-1}, and we conclude that

$$r_n = r_{mk+k-1} = a^m r_{k-1} + b \sum_{i=0}^{m-1} a^i, \quad m = 1, 2, 3, \cdots .$$

Therefore, the general solution of Eq.(4.23) is

$$r_n = r_{nk+t} = a^n r_t + b\frac{a^n - 1}{a - 1} = a^n \left(r_t + \frac{b}{a-1} \right) - \frac{b}{a-1}, \tag{4.24}$$

where $n = 1, 2, 3, \ldots, \forall t = 0, 1, \cdots, k-1$. Now the solution of Eq.(4.21) is

$$x_n = x_0 \prod_{j=1}^{n} r_j = x_0 \prod_{j=1}^{n} \left(a^j \left(r_t + \frac{b}{a-1} \right) - \frac{b}{a-1} \right),$$

$\forall t = 0, \cdots, k-1$ and $r_n = \frac{x_n}{x_{n-1}}$.

Example 4.11. [5] Consider the autonomous rational difference equation of **order** $k+1$

$$x_{n+1} = x_n + \frac{b}{a + x_{n-j} - x_{n-k}}, \tag{4.25}$$

with initial conditions $x_0 > x_{-1} > \cdots > x_{-k}$ where $a, b > 0$, $k \geq 1$ and $0 \leq j \leq k-1$.

Eq.(4.25) is $AH1$, so we can reduce its order by one to have the following equation of order k

$$r_{n+1} = \frac{b}{a + r_{n-k+1} + r_{n-k+2} + \cdots + r_{n-j}}, \tag{4.26}$$

since we substitue $0 \longrightarrow x_n$ and $r_{n-k+1} + r_{n-k+2} + \cdots + r_{n-j} \longrightarrow x_{n-j} - x_{n-k}$, because

$$x_{n-j} = -r_{n-j+1} - r_{n-j+2} - \cdots - r_{n-1} - r_n,$$

and note that the number of terms in x_{n-j} is j.

And since $0 \leq j \leq k-1$,

$$x_{n-k} = -r_{n-k+1} - r_{n-k+2} - \cdots - r_{n-j} - r_{n-j+1} - \cdots - r_{n-1} - r_n,$$

and note the the number of terms in x_{n-k} is k.

Therefore,

$$x_{n-j} - x_{n-k} = r_{n-k+1} + r_{n-k+2} + \cdots + r_{n-j},$$

with $k - j$ terms.

Also since $r_n = x_n - x_{n-1}$ and according to the inital conditions of Eq.(4.25), we have $r_0, r_{-1}, \cdots, r_{-k+1} > 0$ as initial conditions of Eq.(4.26). This implies that $r_n > 0$ for all $n \geq 1$, so the corresponding solution of Eq.(4.25) is increasing and eventually positive since by

$$x_n = x_0 + \sum_{i=1}^{n} r_i,$$

We can transform Eq.(4.26) to a more familiar equation by substituting $t_n = \frac{b}{r_n}$ we have

$$\begin{aligned}
\frac{b}{r_{n+1}} &= a + r_{n-j} + r_{n-j-1} + \cdots + r_{n-k+1}, \\
t_{n+1} &= a + \frac{b}{t_{n-j}} + \frac{b}{t_{n-j-1}} + \frac{b}{t_{n-k+1}}, \\
&= a + b\sum_{i=j}^{k-1} \frac{1}{t_{n-i}}.
\end{aligned}$$

5 ON LIE SYMMETRIES AND DYNAMICS FOR HOMOGENEOUS DIFFERENCE EQUATIONS

In this chapter, we will review an Euler's theorem for multiplicative homogeneous functions and a lemma for additive homogeneous functions. We try to find the characteristic function $Q(n, x_n)$ for some difference equations using Lie symmetry method. Then, we present two theorems that gives the characteristic function $Q(n, x_n)$ directly for $MH1$ and $AH1$ respectively. We do that by using Lie symmetry method and order reduction theorem. Finally, we will review stability to study convergence for $MH1$ of second order.

5.1 *Euler's Theorem For Homogeneous Functions*

In this section, we present another way to check if a given function is homogeneous of degree k rather than the definition.

Theorem 5.1. *[9] Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be continuous, and differentiable on $\mathbb{R}^n_{++}$. Then f is MHk iff for all $x \in \mathbb{R}^n_{++}$,*

$$kf(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i. \tag{5.1}$$

Proof. Suppose f is homogeneous of degree k. Fix $x \in \mathbb{R}^n_{++}$, and define the function $g : [0, \infty) \longrightarrow \mathbb{R}$ (depending on x) by

$$g(t) = f(tx) - t^k f(x),$$

and note that for all $t \geq 0$,

$$g(t) = 0.$$

Therefore, for all $t > 0$,

$$g'(t) = 0.$$

But by the chain rule, since $x \in R^n_{++}$,

$$g'(t) = \sum_{i=1}^{n} \frac{\partial f(tx)}{\partial x_i} x_i - kt^{k-1} f(x) = 0,$$

evaluate $g'(t)$ at $t = 1$ to obtain Eq.(5.1).

Conversely, suppose

$$kf(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i.$$

for all $x \in \mathbb{R}^n_{++}$. Fix any $x \gg 0$ and again define $g : [0, \infty) \longrightarrow \mathbb{R}$ (depending on x) by

$$g(t) = f(tx) - t^k f(x),$$

(need to show g is identically zero) and note that $g(1) = 0$. Then $\forall t > 0$,

$$\begin{aligned} g'(t) &= \sum_{i=1}^{n} \frac{\partial f(tx)}{\partial x_i} x_i - kt^{k-1} f(x) \\ &= t^{-1} \left(\sum_{i=1}^{n} \frac{\partial f(tx)}{\partial x_i} tx_i \right) - kt^{k-1} f(x) \\ &= t^{-1} kf(tx) - kt^{k-1} f(x). \end{aligned}$$

So,

$$\begin{aligned} tg'(t) &= k(f(tx) - t^k f(x)) \\ &= kg(t). \end{aligned}$$

Since t is an arbitrary constant, g satisfies the following differential equation with an inital condition $g(1) = 0$.

$$g'(t) - \frac{k}{t} g(t) = 0.$$

We can solve it easily by separation of variables as follows:

$$\frac{dg(t)}{g(t)} = k\frac{dt}{t},$$

integrating both sides, we get

$$\ln g(t) = k \ln t + c = \ln t^k + c, t > 0.$$

Now apply the natural exponential to have

$$g(t) = Ct^k,$$

where C is an arbitrary constant, by using the initial condition we have $C = 0$. Therefore, g is identically zero, so f is homogeneuous of order k on $\mathbb{R}^n_{++}$. Continuity gurantees that f is homogeneuous on $\mathbb{R}^n_{++}$. □

Lemma 5.2. Let $f : \mathbb{D} \subset \mathbb{R}^k_+ \to \mathbb{R}$ be a class $\mathbb{C}^1$, then f is $AH1$ iff

$$\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_k) = 1. \tag{5.2}$$

Proof. Assume that f is $AH1$; that is, $\forall t \in \mathbb{R}$ and $(x_1, x_2, \dots, x_k) \in \mathbb{D}$,

$$f(t + x_1, t + x_2, \dots, t + x_k) = t + f(x_1, x_2, \dots, x_k).$$

Differentiating both sides with respect to t as follows,

$$\frac{\partial f}{\partial(t+x_1)}\frac{\partial(t+x_1)}{\partial t}+\frac{\partial f}{\partial(t+x_2)}\frac{\partial(t+x_2)}{\partial t}+\cdots+\frac{\partial f}{\partial(t+x_k)}\frac{\partial(t+x_k)}{\partial t}=1+0=1.$$

Clearly, $\frac{\partial(t+x_i)}{\partial t}=1$, $\forall i=1,\ldots,k$ and substitute $t=0$ we get,

$$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial x_2}+\cdots+\frac{\partial f}{\partial x_k}=\sum_{i=1}^{k}\frac{\partial f}{\partial x_i}=1.$$

Conversely, suppose that Eq.(5.2) holds. For all $x\in\mathbb{D}$, define

$$\phi(t)=f(t+x_1,t+x_2,\ldots,t+x_k)-t-f(x_1,x_2,\ldots,x_k)$$

we need to show $\phi(t)=0,\forall t\in\mathbb{R}$. Clearly $\phi(0)=0$, and

$$\phi'(t)=\sum_{i=1}^{k}\frac{\partial f}{\partial(t+x_i)}(t+x_1,t+x_2,\ldots,t+x_k)-1$$

and by Eq.(5.2) for all $(t+x_1,t+x_2,\ldots,t+x_k)\in\mathbb{D}$,

$$\phi'(t)=0.$$

Therefore, $\phi(t)=c$ and since $\phi(0)=0$, we have $c=0$ and hence

$$\phi(t)=0,\ \forall t\in\mathbb{R}.$$

The proof is complete. □

5.2 *Characteristic Function For Some Difference Equations Using Lie Symmetry*

In this section, we try to find a characteristic function Q for some difference equations with different orders by following steps in Lie symmetry method.

5.2.1 The Characteristic Function for the difference 2^{nd} OΔE $x_{n+1} = \frac{x_n^2+x_{n-1}^2}{x_n+x_{n-1}}$

In this section, we need to find the characteristic function for the second order difference equation that is $MH1$,

$$x_{n+1} = \frac{x_n^2 + x_{n-1}^2}{x_n + x_{n-1}} = f(x_n, x_{n-1}). \tag{5.3}$$

To find $Q(n, x_n)$, we write the LSC for Eq.(5.3), as follows

$$Q(n+1, f) - \frac{\partial f}{\partial x_n} Q(n, x_n) - \frac{\partial f}{\partial x_{n-1}} Q(n-1, x_{n-1}) = 0, \tag{5.4}$$

but,

$$\frac{\partial f}{\partial x_n} = \frac{2x_n - f}{x_n + x_{n-1}},$$

$$\frac{\partial f}{\partial x_{n-1}} = \frac{2x_{n-1} - f}{x_n + x_{n-1}},$$

and

$$\frac{\partial x_{n-1}}{\partial x_n} = -\frac{\partial f/\partial x_n}{\partial f/\partial x_{n-1}} = -\frac{2x_n - f}{2x_{n-1} - f},$$

Then Eq.(5.4) becomes

$$Q(n+1, f) - \frac{2x_n - f}{x_n + x_{n-1}} Q(n, x_n) - \frac{2x_{n-1} - f}{x_n + x_{n-1}} Q(n-1, x_{n-1}) = 0, \tag{5.5}$$

apply the differential operator L to Eq.(5.5), where L is defined by

$$L = \frac{\partial}{\partial x_n} + \frac{\partial x_{n-1}}{\partial x_n}\frac{\partial}{\partial x_{n-1}} = \frac{\partial}{\partial x_n} - \left(\frac{2x_n - f}{2x_{n-1} - f}\right)\frac{\partial}{\partial x_{n-1}},$$

$$\begin{aligned}\frac{\partial}{\partial x_n}&\left(Q(n+1, f) - \frac{2x_n - f}{x_n + x_{n-1}} Q(n, x_n) - \frac{2x_{n-1} - f}{x_n + x_{n-1}} Q(n-1, x_{n-1})\right) \\ &- \left(\frac{2x_n - f}{2x_{n-1} - f}\right)\frac{\partial}{\partial x_{n-1}}\left(Q(n+1, f) - \frac{2x_n - f}{x_n + x_{n-1}} Q(n, x_n)\right. \\ &\left. - \frac{2x_{n-1} - f}{x_n + x_{n-1}} Q(n-1, x_{n-1})\right) = 0,\end{aligned} \tag{5.6}$$

but,

$$\frac{\partial}{\partial x_n}(Q(n+1,f)) = 0,$$
$$\frac{\partial}{\partial x_n}\left(\frac{2x_n - f}{x_n + x_{n-1}}Q(n,x_n)\right) = \frac{2x_n - f}{x_n + x_{n-1}}Q'(n,x_n) + \frac{2x_{n-1}+f}{(x_n + x_{n-1})^2}Q(n,x_n),$$
$$\frac{\partial}{\partial x_n}\left(\frac{2x_{n-1} - f}{x_n + x_{n-1}}Q(n-1,x_{n-1})\right) = -\frac{2x_{n-1} - f}{(x_n + x_{n-1})^2}Q(n-1,x_{n-1}),$$

and

$$\frac{\partial}{\partial x_{n-1}}(Q(n+1,f)) = 0,$$
$$\frac{\partial}{\partial x_{n-1}}\left(\frac{2x_n - f}{x_n + x_{n-1}}Q(n,x_n)\right) = -\frac{2x_n - f}{(x_n + x_{n-1})^2}Q(n,x_n),$$
$$\frac{\partial}{\partial x_{n-1}}\left(\frac{2x_{n-1} - f}{x_n + x_{n-1}}Q(n-1,x_{n-1})\right) = \frac{2x_{n-1} - f}{x_n + x_{n-1}}Q'(n-1,x_{n-1}) + \frac{2x_n + f}{(x_n + x_{n-1})^2}Q(n-1,x_{n-1}).$$

This leads Eq.(5.6) to

$$-\frac{2x_n - f}{x_n + x_{n-1}}Q'(n,x_n) - \frac{2x_{n-1}+f}{(x_n + x_{n-1})^2}Q(n,x_n) + \frac{2x_{n-1} - f}{(x_n + x_{n-1})^2}Q(n-1,x_{n-1}) - \left(\frac{2x_n - f}{2x_{n-1} - f}\right)$$
$$\left(\frac{2x_n - f}{(x_n + x_{n-1})^2}Q(n,x_n)) - \frac{2x_{n-1} - f}{x_n + x_{n-1}}Q'(n-1,x_{n-1}) - \frac{2x_n + f}{(x_n + x_{n-1})^2}Q(n-1,x_{n-1})\right) = 0. \tag{5.7}$$

Rearranging and gathering similar terms, we get

$$\frac{2x_n - f}{x_n + x_{n-1}}Q'(n,x_n) + \frac{4x_n^2 + 4x_{n-1}^2 - 4fx_n}{(2x_{n-1} - f)(x_n + x_{n-1})^2}Q(n,x_n) +$$
$$\frac{4fx_{n-1} - 4x_n^2 - 4x_{n-1}^2}{(2x_{n-1} - f)(x_n + x_{n-1})^2}Q(n-1,x_{n-1}) - \frac{2x_n - f}{x_n + x_{n-1}}Q'(n-1,x_{n-1}) = 0, \tag{5.8}$$

multiply Eq.(5.8) by $(2x_{n-1} - f)(x_n + x_{n-1})^2$, we get

$$(2x_n - f)(2x_{n-1} - f)(x_n + x_{n-1})Q'(n, x_n) + (4x_n^2 + 4x_{n-1}^2 - 4fx_n)Q(n, x_n)+ (4fx_{n-1} - 4x_n^2 - 4x_{n-1}^2)Q(n-1, x_{n-1}) - (2x_n - f)(2x_{n-1} - f)(x_n + x_{n-1})Q'(n-1, x_{n-1}) = 0, \quad (5.9)$$

differentiate Eq.(5.9) with respect to x_n yields

$$(2x_n - f)(2x_{n-1} - f)(x_n + x_{n-1})Q''(n, x_n) + ((2x_{n-1} - f)(4x_n + 2x_{n-1} - f)+ 4(x_n^2 + x_{n-1}^2 - fx_n))Q'(n, x_n) + 4(2x_n - f)Q(n, x_n) - 8x_nQ(n-1, x_{n-1}) - ((2x_{n-1} - f)(4x_n + 2x_{n-1} - f))Q'(n-1, x_{n-1}) = 0, \quad (5.10)$$

differentiate Eq.(5.10) with respect to x_n,

$$(2x_n - f)(2x_{n-1} - f)(x_n + x_{n-1})Q'''(n, x_n) + ((2x_{n-1} - f)(4x_n + 2x_{n-1} - f)+ 4(x_n^2 + x_{n-1}^2 - fx_n))Q''(n, x_n) + (16x_n + 8x_{n-1} - 12f)Q'(n, x_n) + 8Q(n, x_n) - 8Q(n-1, x_{n-1}) - 4(2x_{n-1} - f)Q'(n-1, x_{n-1}) = 0, \quad (5.11)$$

differentiate Eq.(5.11) with respect to x_n,

$$(2x_n - f)(2x_{n-1} - f)(x_n + x_{n-1})Q^{(4)}(n, x_n) + ((2x_{n-1} - f)(4x_n + 2x_{n-1} - f)+ 4(x_n^2 + x_{n-1}^2 - fx_n))Q'''(n, x_n) + (24x_n + 16x_{n-1} - 20f)Q''(n, x_n) + 24Q'(n, x_n) = 0, \quad (5.12)$$

differentiate Eq.(5.12) with respect to x_n,

$$(2x_n - f)(2x_{n-1} - f)(x_n + x_{n-1})Q^{(5)}(n, x_n) + ((2x_{n-1} - f)(4x_n + 2x_{n-1} - f)+ 4(x_n^2 + x_{n-1}^2 - fx_n))Q^{(4)}(n, x_n) + (32x_n + 24x_{n-1} - 28f)Q'''(n, x_n) + 48Q''(n, x_n) = 0, \quad (5.13)$$

If we assume $Q(n, x_n) = cx_n$ of Eq.(5.13) where c is a constant, then the characteristic of Eq.(5.3) is $Q(n, x_n) = cx_n$.

5.2.2 The Characteristic Function for the 3^{rd} order difference equation $x_{n+1} = x_n + \frac{a(x_n - x_{n-1})^2}{x_{n-1} - x_{n-2}}$

In this section, we need to find the characteristic function $Q(n, x_n)$, for $MH1$ and $AH1$ difference equation of order 3,

$$x_{n+1} = x_n + \frac{a(x_n - x_{n-1})^2}{x_{n-1} - x_{n-2}} = f(x_n, x_{n-1}, x_{n-2}); \tag{5.14}$$

To find $Q(n, x_n)$, we write LSC for Eq.(5.14), as follows :

$$Q(n+1, f) - \frac{\partial f}{\partial x_n} Q(n, x_n) - \frac{\partial f}{\partial x_{n-1}} Q(n-1, x_{n-1}) - \frac{\partial f}{\partial x_{n-2}} Q(n-2, x_{n-2}) = 0, \tag{5.15}$$

but,

$$\frac{\partial f}{\partial x_n} = 1 + \frac{2a(x_n - x_{n-1})}{x_{n-1} - x_{n-2}} = 1 + \frac{2(f - x_n)}{x_n - x_{n-1}} = \frac{2f - x_{n-1} - x_n}{x_n - x_{n-1}},$$

$$\frac{\partial f}{\partial x_{n-1}} = (f - x_n)\left(\frac{2}{x_{n-1} - x_n} + \frac{1}{x_{n-2} - x_{n-1}}\right),$$

and

$$\frac{\partial f}{\partial x_{n-2}} = \frac{a(x_n - x_{n-1})^2}{(x_{n-1} - x_{n-2})^2} = \frac{f - x_n}{x_{n-1} - x_{n-2}}.$$

so the LSC is

$$Q(n+1, f) + \frac{2f - x_{n-1} - x_n}{x_{n-1} - x_n} Q(n, x_n) - (f - x_n)\left(\frac{2}{x_{n-1} - x_n} + \frac{1}{x_{n-2} - x_{n-1}}\right)$$
$$Q(n-1, x_{n-1}) + \frac{f - x_n}{x_{n-2} - x_{n-1}} Q(n-2, x_{n-2}) = 0, \tag{5.16}$$

Now, we apply the differential operator L, given by

$$L = \frac{\partial}{\partial x_n} + \frac{\partial x_{n-2}}{\partial x_n} \frac{\partial}{\partial x_{n-2}},$$

but,

$$\frac{\partial x_{n-2}}{\partial x_n} = -\frac{\partial f / \partial x_n}{\partial f / \partial x_{n-2}} = -\left(\frac{1 + \frac{2(f - x_n)}{x_n - x_{n-1}}}{\frac{f - x_n}{x_{n-1} - x_{n-2}}}\right) = \frac{(x_{n-2} - x_{n-1})(2f - x_n - x_{n-1})}{(f - x_n)(x_n - x_{n-1})},$$

Now, we apply the differential operator L to Eq.(5.16) to get

$$\frac{\partial}{\partial x_n}\Bigg(Q(n+1,f)+\frac{2f-x_{n-1}-x_n}{x_{n-1}-x_n}Q(n,x_n)-(f-x_n)\Bigg(\frac{2}{x_{n-1}-x_n}+\frac{1}{x_{n-2}-x_{n-1}}\Bigg)$$
$$Q(n-1,x_{n-1})+\frac{f-x_n}{x_{n-2}-x_{n-1}}Q(n-2,x_{n-2})\Bigg)+\Bigg(\frac{(x_{n-2}-x_{n-1})(2f-x_n-x_{n-1})}{(f-x_n)(x_n-x_{n-1})}\Bigg)$$
$$\frac{\partial}{\partial x_{n-2}}\Bigg(Q(n+1,f)+\frac{2f-x_{n-1}-x_n}{x_{n-1}-x_n}Q(n,x_n)-(f-x_n)\Bigg(\frac{2}{x_{n-1}-x_n}+\frac{1}{x_{n-2}-x_{n-1}}\Bigg)$$
$$Q(n-1,x_{n-1})+$$
$$\frac{f-x_n}{x_{n-2}-x_{n-1}}Q(n-2,x_{n-2})\Bigg)=0, \quad (5.17)$$

but

$$\frac{\partial}{\partial x_n}\Bigg(Q(n+1,f)\Bigg)=0,$$
$$\frac{\partial}{\partial x_n}\Bigg(\frac{2f-x_{n-1}-x_n}{x_{n-1}-x_n}Q(n,x_n)\Bigg)=\frac{2f-x_{n-1}-x_n}{x_{n-1}-x_n}Q'(n,x_n)+\frac{2(f-x_{n-1})}{(x_{n-1}-x_n)^2}Q(n,x_n),$$
$$\frac{\partial}{\partial x_n}\Bigg(\frac{f-x_n}{x_{n-2}-x_{n-1}}Q(n-2,x_{n-2})\Bigg)=-\frac{Q(n-2,x_{n-2})}{x_{n-2}-x_{n-1}},$$

$$\frac{\partial}{\partial x_n}\Bigg((f-x_n)\Bigg(\frac{2}{x_{n-1}-x_n}+\frac{1}{x_{n-2}-x_{n-1}}\Bigg)Q(n-1,x_{n-1})\Bigg)=$$
$$\frac{2(f-x_{n-1})}{(x_{n-1}-x_n)^2}Q(n-1,x_{n-1})-\frac{Q(n-1,x_{n-1})}{x_{n-2}-x_{n-1}},$$

and

$$\frac{\partial}{\partial x_{n-2}}\Bigg(Q(n+1,f)\Bigg)=0,$$
$$\frac{\partial}{\partial x_{n-2}}\Bigg(\frac{2f-x_{n-1}-x_n}{x_{n-1}-x_n}Q(n,x_n)\Bigg)=0,$$

$$\frac{\partial}{\partial x_{n-2}}\left((f-x_n)\left(\frac{2}{x_{n-1}-x_n}+\frac{1}{x_{n-2}-x_{n-1}}\right)Q(n-1,x_{n-1})\right) = -\frac{f-x_n}{(x_{n-2}-x_{n-1})^2} Q(n-1,x_{n-1}),$$

$$\frac{\partial}{\partial x_{n-2}}\left(\frac{f-x_n}{x_{n-2}-x_{n-1}}Q(n-2,x_{n-2})\right) = \frac{f-x_n}{x_{n-2}-x_{n-1}}Q'(n-2,x_{n-2})-\frac{f-x_n}{(x_{n-2}-x_{n-1})^2} Q(n-2,x_{n-2}),$$

Substitute the above partial derivatives in Eq.(5.17) and rearrange the result to have

$$\begin{aligned}&\frac{2f-x_{n-1}-x_n}{x_{n-1}-x_n}Q'(n,x_n)+\frac{2(f-x_{n-1})}{(x_n-x_{n-1})^2}Q(n,x_n)\\&+\frac{2(x_{n-1}-f)(x_{n-2}-x_{n-1})^2+(x_{n-1}-x_n)^2(f+x_{n-2}-x_n-x_{n-1})}{(x_{n-1}-x_n)^2(x_{n-2}-x_{n-1})^2}Q(n-1,x_{n-1})\\&+\frac{2(x_{n-1}-f)}{(x_n-x_{n-1})(x_{n-2}-x_{n-1})}Q(n-2,x_{n-2})+\frac{2f-x_{n-1}-x_n}{x_n-x_{n-1}}Q'(n-2,x_{n-2})=0,\end{aligned} \tag{5.18}$$

multiply Eq.(5.18) by $(x_n-x_{n-1})^2$, and then differentiate the result with respect to x_n, we get

$$\begin{aligned}&(x_{n-1}-x_n)(2f-x_{n-1}-x_n)Q''(n,x_n)+2(x_n-x_{n-1})Q'(n,x_n)\\&\quad-\frac{(x_{n-1}-x_n)(2f+2x_{n-2}-3x_n-x_{n-1})}{(x_{n-2}-x_{n-1})^2}Q(n-1,x_{n-1})\\&\quad+\frac{2(x_{n-1}-f)}{x_{n-2}-x_{n-1}}Q(n-2,x_{n-2})+2(f-x_n)Q'(n-2,x_{n-2})=0,\end{aligned} \tag{5.19}$$

differentiate Eq.(5.19) with respect to x_n and rearrange the result to have

$$\begin{aligned}&(x_{n-1}-x_n)(2f-x_{n-1}-x_n)Q^{(3)}(n,x_n)-2(f-2x_n+x_{n-1})Q''(n,x_n)+2Q'(n,x_n)\\&+\frac{2f+2x_{n-2}-6x_n+2x_{n-1}}{(x_{n-2}-x_{n-1})^2}Q(n-1,x_{n-1})-2Q'(n-2,x_{n-2})=0,\end{aligned} \tag{5.20}$$

another time, differentiate Eq.(5.20) with respect to x_n and rearrange the result to have

$$(x_{n-1}-x_n)(2f-x_{n-1}-x_n)Q^{(4)}(n,x_n)-2(2f-3x_n+x_{n-1})Q^{(3)}(n,x_n)+6Q''(n,x_n) - \frac{6}{(x_{n-2}-x_{n-1})^2}Q(n-1,x_{n-1})=0, \quad (5.21)$$

differentiate Eq.(5.21) with respect to x_n and rearrange the result to have

$$(x_{n-1}-x_n)(2f-x_{n-1}-x_n)Q^{(5)}(n,x_n)-2(3f-4x_n+x_{n-1})Q^{(4)}(n,x_n) + 12Q^{(3)}(n,x_n)=0. \quad (5.22)$$

If we assume $Q(n,x_n) = cx_n$ or $Q(n,x_n) = c$, then they are solutions of Eq.(5.22).

Therefore the characteristic function of Eq.(5.14) is $Q(n,x_n) = cx_n$ or $Q(n,x_n) = c$, where c is a constant.

5.2.3 The Characteristic Function for the 4^{th} order difference equation $x_{n+1} = x_n + \frac{a(x_n-x_{n-1})^2}{x_{n-2}-x_{n-3}}$

In this section, we need to find the characteristic function $Q(n,x_n)$, for $MH1$ and $AH1$ difference equation of order 4,

$$x_{n+1} = x_n + \frac{a(x_n-x_{n-1})^2}{x_{n-2}-x_{n-3}} = f(x_n,x_{n-1},x_{n-2},x_{n-3}); \quad (5.23)$$

To find $Q(n,x_n)$, we write LSC for Eq.(5.23), as follows

$$Q(n+1,f) - \frac{\partial f}{\partial x_n}Q(n,x_n) - \frac{\partial f}{\partial x_{n-1}}Q(n-1,x_{n-1}) - \frac{\partial f}{\partial x_{n-2}}Q(n-2,x_{n-2}) - \frac{\partial f}{\partial x_{n-3}}Q(n-3,x_{n-3}) = 0, \quad (5.24)$$

but

$$\frac{\partial f}{\partial x_n} = 1 + \frac{2a(x_n-x_{n-1})}{x_{n-2}-x_{n-3}} = 1 + \frac{2(f-x_n)}{x_n-x_{n-1}},$$

$$\frac{\partial f}{\partial x_{n-1}} = \frac{-2a(x_n - x_{n-1})}{x_{n-2} - x_{n-3}} = \frac{-2(f - x_n)}{x_n - x_{n-1}},$$

$$\frac{\partial f}{\partial x_{n-2}} = \frac{-a(x_n - x_{n-1})^2}{(x_{n-2} - x_{n-3})^2} = \frac{-(f - x_n)}{x_{n-2} - x_{n-3}},$$

and

$$\frac{\partial f}{\partial x_{n-3}} = \frac{a(x_n - x_{n-1})^2}{(x_{n-2} - x_{n-3})^2} = \frac{f - x_n}{x_{n-2} - x_{n-3}}.$$

So the LSC is

$$Q(n+1, f) - \left(1 + \frac{2(f - x_n)}{x_n - x_{n-1}}\right) Q(n, x_n) + \frac{2(f - x_n)}{x_n - x_{n-1}} Q(n-1, x_{n-1}) + \frac{f - x_n}{x_{n-2} - x_{n-3}} Q(n-2, x_{n-2}) - \frac{f - x_n}{x_{n-2} - x_{n-3}} Q(n-3, x_{n-3}) = 0. \quad (5.25)$$

Now, we apply the differential operator L, given by

$$L = \frac{\partial}{\partial x_n} + \frac{\partial x_{n-1}}{\partial x_n} \frac{\partial}{\partial x_{n-1}},$$

but,

$$\frac{\partial x_{n-1}}{\partial x_n} = -\frac{\partial f/\partial x_n}{\partial f/\partial x_{n-1}} = -\left(\frac{1 + \frac{2(f - x_n)}{x_n - x_{n-1}}}{\frac{-2(f - x_n)}{x_n - x_{n-1}}}\right) = \frac{2f - x_n - x_{n-1}}{2(f - x_n)}.$$

Now, we apply the differential operator L to Eq.(5.25) to get

$$\frac{\partial}{\partial x_n}\left(Q(n+1, f) - \left(1 + \frac{2(f - x_n)}{x_n - x_{n-1}}\right) Q(n, x_n) + \frac{2(f - x_n)}{x_n - x_{n-1}} Q(n-1, x_{n-1}) + \frac{f - x_n}{x_{n-2} - x_{n-3}} Q(n-2, x_{n-2}) - \frac{f - x_n}{x_{n-2} - x_{n-3}} Q(n-3, x_{n-3})\right) + \left(\frac{2f - x_n - x_{n-1}}{2(f - x_n)}\right)$$
$$\frac{\partial}{\partial x_{n-1}}\left(Q(n+1, f) - \left(1 + \frac{2(f - x_n)}{x_n - x_{n-1}}\right) Q(n, x_n) + \frac{2(f - x_n)}{x_n - x_{n-1}} Q(n-1, x_{n-1}) + \frac{f - x_n}{x_{n-2} - x_{n-3}} Q(n-2, x_{n-2}) - \frac{f - x_n}{x_{n-2} - x_{n-3}} Q(n-3, x_{n-3})\right) = 0, \quad (5.26)$$

but

$$\frac{\partial}{\partial x_n}(Q(n+1,f)) = 0,$$
$$\frac{\partial}{\partial x_n}\left(\left(1+\frac{2(f-x_n)}{x_n-x_{n-1}}\right)Q(n,x_n)\right) = \frac{2f-x_n-x_{n-1}}{x_n-x_{n-1}}Q'(n,x_n)+\frac{2(x_{n-1}-f)}{(x_n-x_{n-1})^2}Q(n,x_n),$$
$$\frac{\partial}{\partial x_n}\left(\frac{2(f-x_n)}{x_n-x_{n-1}}Q(n-1,x_{n-1})\right) = \frac{2(x_{n-1}-f)}{(x_n-x_{n-1})^2}Q(n-1,x_{n-1}),$$
$$\frac{\partial}{\partial x_n}\left(\frac{f-x_n}{x_{n-2}-x_{n-3}}Q(n-2,x_{n-2})\right) = \frac{-Q(n-2,x_{n-2})}{x_{n-2}-x_{n-3}},$$
$$\frac{\partial}{\partial x_n}\left(\frac{f-x_n}{x_{n-2}-x_{n-3}}Q(n-3,x_{n-3})\right) = \frac{-Q(n-3,x_{n-3})}{x_{n-2}-x_{n-3}},$$

and

$$\frac{\partial}{\partial x_{n-1}}(Q(n+1,f)) = 0,$$
$$\frac{\partial}{\partial x_{n-1}}\left(\left(1+\frac{2(f-x_n)}{x_n-x_{n-1}}\right)Q(n,x_n)\right) = \frac{2(f-x_n)}{(x_n-x_{n-1})^2}Q(n,x_n),$$
$$\frac{\partial}{\partial x_{n-1}}\left(\frac{2(f-x_n)}{x_n-x_{n-1}}Q(n-1,x_{n-1})\right) = \frac{2(f-x_n)}{x_n-x_{n-1}}Q'(n-1,x_{n-1})+\frac{2(f-x_n)}{(x_n-x_{n-1})^2}Q(n-1,x_{n-1}),$$
$$\frac{\partial}{\partial x_{n-1}}\left(\frac{f-x_n}{x_{n-2}-x_{n-3}}Q(n-2,x_{n-2})\right) = 0,$$
$$\frac{\partial}{\partial x_{n-1}}\left(\frac{f-x_n}{x_{n-2}-x_{n-3}}Q(n-3,x_{n-3})\right) = 0.$$

Substitute the above partial derivatives in Eq.(5.26) to have

$$-\frac{2f-x_n-x_{n-1}}{x_n-x_{n-1}}Q'(n,x_n)-\frac{2(x_{n-1}-f)}{(x_n-x_{n-1})^2}Q(n,x_n)+\frac{2(x_{n-1}-f)}{(x_n-x_{n-1})^2}Q(n-1,x_{n-1})$$
$$-\frac{Q(n-2,x_{n-2})}{x_{n-2}-x_{n-3}}+\frac{Q(n-3,x_{n-3})}{x_{n-2}-x_{n-3}}+\left(\frac{2f-x_n-x_{n-1}}{2(f_n-x_n)}\right)\left(-\frac{2(f-x_n)}{(x_n-x_{n-1})^2}Q(n,x_n)\right.$$
$$\left.+\frac{2(f-x_n)}{x_n-x_{n-1}}Q'(n-1,x_{n-1})+\frac{2(f-x_n)}{(x_n-x_{n-1})^2}Q(n-1,x_{n-1})\right) = 0,$$

which can be writen as

$$\frac{x_n + x_{n-1} - 2f}{x_n - x_{n-1}}Q'(n, x_n) + \frac{Q(n, x_n)}{x_n - x_{n-1}} + \frac{Q(n-1, x_{n-1})}{x_{n-1} - x_n} + \frac{Q(n-2, x_{n-2})}{x_{n-3} - x_{n-2}} + \frac{Q(n-3, x_{n-3})}{x_{n-2} - x_{n-3}} + \frac{2f - x_n - x_{n-1}}{x_n - x_{n-1}}Q'(n-1, x_{n-1}) = 0, \quad (5.27)$$

we differentiate Eq.(5.27) with respect to x_n to get

$$\frac{x_n + x_{n-1} - 2f}{x_n - x_{n-1}}Q''(n, x_n) + \frac{2f + x_n - 3x_{n-1}}{(x_n - x_{n-1})^2}Q'(n, x_n) - \frac{Q(n, x_n)}{(x_n - x_{n-1})^2} + \frac{Q(n-1, x_{n-1})}{(x_{n-1} - x_n)^2} + \frac{2(x_{n-1} - f)}{(x_n - x_{n-1})^2}Q'(n-1, x_{n-1}) = 0, \quad (5.28)$$

multiply Eq.(5.28) by $(x_n - x_{n-1})^2$, we get

$$(x_n^2 - x_{n-1}^2 - 2fx_n + 2fx_{n-1})Q''(n, x_n) + (2f + x_n - 3x_{n-1})Q'(n, x_n) - Q(n, x_n) + Q(n-1, x_{n-1}) + 2(x_{n-1} - f)Q'(n-1, x_{n-1}) = 0, \quad (5.29)$$

differentiate Eq.(5.29) with respect to x_n, we get

$$(x_n^2 - x_{n-1}^2 - 2fx_n + 2fx_{n-1})Q'''(n, x_n) + 3(x_n - x_{n-1})Q''(n, x_n) = 0, \quad (5.30)$$

divide Eq.(5.30) over $x_n - x_{n-1}$ we get

$$(x_n + x_{n-1} - 2f)Q'''(n, x_n) + 3Q''(n, x_n) = 0. \quad (5.31)$$

If we assume $Q(n, x_n) = cx_n$ or $Q(n, x_n) = c$, then they are solutions of Eq.(5.31).

Therefore, the characteristic function of Eq.(5.23) is $Q(n, x_n) = cx_n$ or $Q(n, x_n) = c$, where c is a constant.

5.2.4 The Characteristic Function for the difference equation $x_{n+1} = \frac{x_{n-l}x_{n-k}}{ax_{n-k}+bx_{n-l}}$ of order $k+1$

We need to find $Q(n, x_n)$ for the following $MH1$ difference equation of order $k+1$

$$x_{n+1} = \frac{x_{n-l}x_{n-k}}{ax_{n-k} + bx_{n-l}} = f(x_n, x_{n-1}, \ldots, x_{n-l}, \ldots, x_{n-k}); \quad (5.32)$$

where $k > l$.

The LSC of Eq.(5.32) is

$$Q(n+1,f)-\frac{\partial f}{\partial x_n}Q(n,x_n)-\frac{\partial f}{\partial x_{n-1}}Q(n-1,x_{n-1})-\cdots-\frac{\partial f}{\partial x_{n-l}}Q(n-l,x_{n-l})-\cdots-\frac{\partial f}{\partial x_{n-k}}Q(n-k,x_{n-k})=0, \quad (5.33)$$

but

$$\frac{\partial f}{\partial x_{n-l}} = \frac{af^2}{x^2_{n-l}},$$
$$\frac{\partial f}{\partial x_{n-k}} = \frac{bf^2}{x^2_{n-k}},$$

and

$$\frac{\partial f}{\partial x_{n-i}} = 0, \forall i \neq l, k.$$

Now, substitute the above partial derivatives in Eq.(5.33) we get

$$Q(n+1,f) - \frac{af^2}{x^2_{n-l}}Q(n-l,x_{n-l}) - \frac{bf^2}{x^2_{n-k}}Q(n-k,x_{n-k}) = 0, \quad (5.34)$$

applying the differential operator L to Eq.(5.34) such that

$$L = \frac{\partial}{\partial x_{n-l}} + \frac{\partial x_{n-k}}{\partial x_{n-l}}\frac{\partial}{\partial x_{n-k}},$$

but

$$\frac{\partial x_{n-k}}{\partial x_{n-l}} = -\frac{\partial f/\partial x_{n-l}}{\partial f/\partial x_{n-k}} = -\frac{ax^2_{n-k}}{bx^2_{n-l}},$$

we have

$$L = \frac{\partial}{\partial x_{n-l}} - \frac{ax^2_{n-k}}{bx^2_{n-l}}\frac{\partial}{\partial x_{n-k}}.$$

We have

$$\frac{\partial}{\partial x_{n-l}}\left(Q(n+1,f) - \frac{af^2}{x^2_{n-l}}Q(n-l,x_{n-l}) - \frac{bf^2}{x^2_{n-k}}Q(n-k,x_{n-k})\right)$$
$$-\frac{ax^2_{n-k}}{bx^2_{n-l}}\frac{\partial}{\partial x_{n-k}}\left(Q(n+1,f)-\frac{af^2}{x^2_{n-l}}Q(n-l,x_{n-l})-\frac{bf^2}{x^2_{n-k}}Q(n-k,x_{n-k})\right) = 0, \quad (5.35)$$

but

$$\frac{\partial}{\partial x_{n-l}}\left(Q(n+1,f)\right)=0,$$
$$\frac{\partial}{\partial x_{n-l}}\left(\frac{af^2}{x_{n-l}^2}Q(n-l,x_{n-l})\right)=\frac{af^2}{x_{n-l}^2}Q'(n-l,x_{n-l})-\frac{2af^2}{x_{n-l}^3}Q(n-l,x_{n-l}),$$
$$\frac{\partial}{\partial x_{n-l}}\left(\frac{bf^2}{x_{n-k}^2}Q(n-k,x_{n-k})\right)=0,$$

and

$$\frac{\partial}{\partial x_{n-k}}\left(Q(n+1,f)\right)=0,$$
$$\frac{\partial}{\partial x_{n-k}}\left(\frac{af^2}{x_{n-l}^2}Q(n-l,x_{n-l})\right)=0,$$
$$\frac{\partial}{\partial x_{n-k}}\left(\frac{bf^2}{x_{n-k}^2}Q(n-k,x_{n-k})\right)=\frac{bf^2}{x_{n-k}^2}Q'(n-k,x_{n-k})-\frac{2bf^2}{x_{n-k}^3}Q(n-k,x_{n-k}).$$

Substitute the above partial derivatives in Eq.(5.35) to get

$$\frac{-af^2}{x_{n-l}^2}Q'(n-l,x_{n-l})+\frac{2af^2}{x_{n-l}^3}Q(n-l,x_{n-l})+\frac{af^2}{x_{n-l}^2}Q'(n-k,x_{n-k})-\frac{2af^2}{x_{n-l}^2x_{n-k}}Q(n-k,x_{n-k})=0, \tag{5.36}$$

multiply Eq.(5.36) by $\frac{-x_{n-l}^2}{af^2}$ we get

$$Q'(n-l,x_{n-l})-\frac{2}{x_{n-l}}Q(n-l,x_{n-l})-Q'(n-k,x_{n-k})+\frac{2}{x_{n-k}}Q(n-k,x_{n-k})=0, \tag{5.37}$$

differentiate Eq.(5.37) with respect to x_{n-l} to have

$$Q''(n-l,x_{n-l})-\frac{2}{x_{n-l}}Q'(n-l,x_{n-l})+\frac{2}{x_{n-l}^2}Q(n-l,x_{n-l})=0, \tag{5.38}$$

multiply Eq.(5.38) by x_{n-l}^2 we get

$$x_{n-l}^2Q''(n-l,x_{n-l})-2x_{n-l}Q'(n-l,x_{n-l})+2Q(n-l,x_{n-l})=0. \tag{5.39}$$

The above second order differential equation is Euler equation we solve it as follows.

Assume the solution is

$$Q(n-l, x_{n-l}) = x_{n-l}^{r}.$$

Then

$$Q'(n-l, x_{n-l}) = r x_{n-l}^{r-1}$$

and

$$Q''(n-l, x_{n-l}) = r(r-1) x_{n-l}^{r-2}.$$

Substitute Q, Q' and Q'' in Eq.(5.39), we get:

$$(r^2 - r)x_{n-l}^{r} - 2r x_{n-l}^{r} + 2x_{n-l}^{r} = 0,$$

$$(r^2 - 3r + 2)x_{n-l}^{r} = 0,$$

$$(r^2 - 3r + 2) = 0,$$

we have $r = 1$ or $r = 2$. Then the characteristic solution of Eq.(5.32)is

$$Q(n-l, x_{n-l}) = c_1 x_{n-l} + c_2 x_{n-l}^{2},$$

where c_1 and c_2 are constants.

5.2.5 The Characteristic Function of the difference equation $x_{n+1} = \frac{x_n x_{n-k}}{a x_{n-k} + b x_{n-l}}$ of order $k+1$

We need to find $Q(n, x_n)$ for the following $MH1$ difference equation of order $k+1$

$$x_{n+1} = \frac{x_n x_{n-k}}{a x_{n-k} + b x_{n-l}} = f(x_n, x_{n-1}, \ldots, x_{n-l}, \ldots, x_{n-k}); \tag{5.40}$$

where $k > l$ The LSC of Eq.(5.40) is

$$Q(n+1,f)-\frac{\partial f}{\partial x_n}Q(n,x_n)-\frac{\partial f}{\partial x_{n-1}}Q(n-1,x_{n-1})-\cdots-\frac{\partial f}{\partial x_{n-l}}Q(n-l,x_{n-l})-\cdots-\frac{\partial f}{\partial x_{n-k}}Q(n-k,x_{n-k})=0, \quad (5.41)$$

but

$$\frac{\partial f}{\partial x_n}=\frac{f}{x_n}$$

$$\frac{\partial f}{\partial x_{n-l}}=\frac{-bf^2}{x_n x_{n-k}},$$
$$\frac{\partial f}{\partial x_{n-k}}=\frac{bf^2 x_{n-l}}{x_n x_{n-k}^2},$$

and

$$\frac{\partial f}{\partial x_{n-i}}=0, \forall i\neq 0,l,k.$$

Now, substitute the above partial derivatives in Eq.(5.41) we get

$$Q(n+1,f)-\frac{f}{x_n}Q(n,x_n)+\frac{bf^2}{x_n x_{n-k}}Q(n-l,x_{n-l})-\frac{bf^2 x_{n-l}}{x_n x_{n-k}^2}Q(n-k,x_{n-k})=0, \quad (5.42)$$

applying the differential operator L to Eq.(5.42) such that

$$L=\frac{\partial}{\partial x_n}+\frac{\partial x_{n-l}}{\partial x_n}\frac{\partial}{\partial x_{n-l}}=\frac{\partial}{\partial x_n}+\frac{x_{n-k}}{bf}\frac{\partial}{\partial x_{n-l}},$$

such that

$$\frac{\partial x_{n-l}}{\partial x_n}=-\frac{\partial f/\partial x_n}{\partial f/\partial x_{n-l}}=\frac{x_{n-k}}{bf}.$$

we have

$$\frac{\partial}{\partial x_n}\left(Q(n+1,f)-\frac{f}{x_n}Q(n,x_n)+\frac{bf^2}{x_n x_{n-k}}Q(n-l,x_{n-l})-\frac{bf^2 x_{n-l}}{x_n x_{n-k}^2}Q(n-k,x_{n-k})\right)$$
$$+\frac{x_{n-k}}{bf}\frac{\partial}{\partial x_{n-l}}\left(Q(n+1,f)-\frac{f}{x_n}Q(n,x_n)+\frac{bf^2}{x_n x_{n-k}}Q(n-l,x_{n-l})-\frac{bf^2 x_{n-l}}{x_n x_{n-k}^2}Q(n-k,x_{n-k})\right)=0, \quad (5.43)$$

but

$$\frac{\partial}{\partial x_n}\left(Q(n+1,f)\right)=0,$$
$$\frac{\partial}{\partial x_n}\left(\frac{f}{x_n}Q(n,x_n)\right)=\frac{f}{x_n}Q'(n,x_n)-\frac{f}{x_n^2}Q(n,x_n),$$
$$\frac{\partial}{\partial x_n}\left(\frac{bf^2}{x_nx_{n-k}}Q(n-l,x_{n-l})\right)=\frac{-bf^2}{x_n^2x_{n-k}}Q(n-l,x_{n-l}),$$
$$\frac{\partial}{\partial x_n}\left(\frac{bf^2x_{n-l}}{x_nx_{n-k}^2}Q(n-k,x_{n-k})\right)=\frac{-bf^2x_{n-l}}{x_n^2x_{n-k}^2}Q(n-k,x_{n-k}),$$

and

$$\frac{\partial}{\partial x_{n-l}}\left(Q(n+1,f)\right)=0,$$
$$\frac{\partial}{\partial x_{n-l}}\left(\frac{f}{x_n}Q(n,x_n)\right)=0,$$
$$\frac{\partial}{\partial x_{n-l}}\left(\frac{bf^2}{x_nx_{n-k}}Q(n-l,x_{n-l})\right)=\frac{bf^2}{x_nx_{n-k}}Q'(n-l,x_{n-l}),$$
$$\frac{\partial}{\partial x_{n-l}}\left(\frac{bf^2x_{n-l}}{x_nx_{n-k}^2}Q(n-k,x_{n-k})\right)=\frac{bf^2}{x_nx_{n-k}^2}Q(n-k,x_{n-k}).$$

Substitute the above partial derivatives in Eq.(5.43) to get

$$\frac{-f}{x_n}Q'(n,x_n)+\frac{f}{x_n^2}Q(n,x_n)-\frac{bf^2}{x_n^2x_{n-k}}Q(n-l,x_{n-l})+\frac{f}{x_n}Q'(n-l,x_{n-l})+ \frac{f(bfx_{n-l}-x_nx_{n-k})}{x_n^2x_{n-k}^2}Q(n-k,x_{n-k})=0, \quad (5.44)$$

multiply Eq.(5.44) by $\frac{-x_n^2}{f}$ to get

$$x_nQ'(n,x_n)-Q(n,x_n)+\frac{bf}{x_{n-k}}Q(n-l,x_{n-l})-x_nQ'(n-l,x_{n-l})- \frac{(bfx_{n-l}-x_nx_{n-k})}{x_{n-k}^2}Q(n-k,x_{n-k})=0, \quad (5.45)$$

differentiate Eq.(5.45) twice with respect to x_n, we get

$$x_n Q'''(n, x_n) + Q''(n, x_n) = 0, \tag{5.46}$$

If we assume $Q(n, x_n) = cx_n$, then $Q(n, x_n) = cx_n$ or $Q(n, x_n) = c$ are solutions of Eq.(5.46).

Therefore the characteristic function of Eq.(5.40) is $Q(n, x_n) = cx_n$, or $Q(n, x_n) = c$, where c is a constant.

5.3 *On Lie Symmetries Of Homogeneous Difference Equations*

In this section, we consier the difference equations of order $k + 1$

$$x_{n+1} = f_n(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}) \quad s.t\ f_n : \mathbb{R}_+^{k+1} \to \mathbb{R}. \tag{5.47}$$

We present theorems that give explicit formulas for the characteristic function Q in $MH1$ and $AH1$ cases. Then we solve Eq.(5.47) using Lie symmerty method, and we reach to the same solution as order reduction theorem for $HD1$.

5.3.1 *On Lie Symmetries Of Multiplicative Homogeneous Difference Equations*

Theorem 5.3. *If the difference equation Eq.(5.47) is $MH1$, then a characteristic function of the Lie symmetry is*

$$Q(n, x_n) = x_n.$$

Proof. Let f_n be $MH1$ for all n. Euler's formula for $MH1$ implies that

$$\frac{\partial f_n}{\partial x_n} x_n + \frac{\partial f_n}{\partial x_{n-1}} x_{n-1} + \cdots + \frac{\partial f_n}{\partial x_{n-k}} x_{n-k} = f_n(x_n, x_{n-1}, \ldots, x_{n-k}),$$

but $f_n(x_n, x_{n-1}, \ldots, x_{n-k}) = x_{n+1} = Q(n+1, x_{n+1})$.

This completes the proof.

□

Now, to solve Eq.(5.47) using Lie symmetry. Firstly, we find the invariant, by solving

$$\frac{dx_n}{x_n} = \frac{dx_{n-1}}{x_{n-1}} = \cdots = \frac{dx_{n-k}}{x_{n-k}} = \frac{dv_n}{0},$$

taking the i^{th} and j^{th} invariants for any $i, j = n-k, \ldots, n$, we solve

$$\frac{dx_i}{x_i} = \frac{dx_j}{x_j},$$

we get

$$\ln x_i = \ln x_j + c_{ij},$$

which gives

$$\frac{x_i}{x_j} = k_{ij},$$

for some constant $k_{ij} \in \mathbb{R}$. And then we solve for $i = n-k, \ldots, n$,

$$\frac{dx_i}{x_i} = \frac{dv}{0},$$

$$\int dv = \int 0,$$

$$v = constant = \psi(k_{n,n-1}, \ldots, k_{n,n-k}, k_{n-1,n}, k_{n-1,n-2}, \ldots, k_{n-1,n-k}, \ldots, k_{n-k,n-k+1}),$$

take $\psi(k_{n,n-1}, \ldots, k_{n,n-k}, k_{n-1,n}, k_{n-1,n-2}, \ldots, k_{n-1,n-k}, \ldots, k_{n-k,n-k+1}) = k_{n,n-1}$, this gives

$$v_n = k_{n,n-1} = \frac{x_n}{x_{n-1}},$$

and this invariant satisfies $Iv_n = 0$, where

$$Iv_n = \sum_{i=0}^{k} Q(n-i, x_{n-i})\frac{\partial v_n}{\partial x_{n-i}},$$

and hence

$$\begin{aligned}
Iv_n &= Q(n,x_n)\frac{\partial(\frac{x_n}{x_{n-1}})}{\partial x_n} + Q(n-1,x_{n-1})\frac{\partial(\frac{x_n}{x_{n-1}})}{\partial x_{n-1}} + \cdots + Q(n-k,x_{n-k})\frac{\partial(\frac{x_n}{x_{n-1}})}{\partial x_{n-k}} \\
&= x_n\frac{1}{x_{n-1}} + x_{n-1}x_n\frac{-1}{x_{n-1}^2} + \cdots + x_{n-k}0 \\
&= 0.
\end{aligned}$$

Now we need to write Eq.(5.47) as an equation of v_n, we use homogeneity of f_n to get

$$\frac{x_{n+1}}{x_n} = f_n(1, \frac{x_{n-1}}{x_n}, \frac{x_{n-2}}{x_n}, \ldots, \frac{x_{n-k}}{x_n}),$$

which can be written as

$$v_{n+1} = f_n(1, v_n^{-1}, (v_{n-1}v_n)^{-1}, \ldots, (v_{n-k+1}...v_{n-1}v_n)^{-1}),$$

which is an equation of order k, we solve it for v_n, after that we use the equation $x_n = v_n x_{n-1}$ to find the solution of the original equation.
The canonical coordinate in this case is given by

$$s_n = \int \frac{dx_n}{x_n} = \ln x_n.$$

It follows that

$$s_{n+1} - s_n = \ln(\frac{x_{n+1}}{x_n}) = \ln v_{n+1},$$

which is a first order difference equation, whose solution is given by

$$s_n = s_0 + \sum_{i=1}^{n} \ln v_i,$$

therefore,

$$x_n = e^{s_n} = e^{s_0}e^{\sum_{i=1}^{n}\ln v_i} = x_0\prod_{i=1}^{n} v_i, \quad n = 1, 2, 3, \ldots.$$

5.3.2 On Lie Symmetries Of Additive Homogeneuous Difference Equations

Theorem 5.4. *If the difference equation Eq.(5.47) is $AH1$, then a characteristic of the Lie group symmetry is*

$$Q(n, x_n) = c, \qquad c \in \mathbb{R}.$$

Proof. Let f_n be $AH1$ for all n, Euler's formula for $AH1$ implies that

$$\frac{\partial f_n}{\partial x_n} + \frac{\partial f_n}{\partial x_{n-1}} + \cdots + \frac{\partial f_n}{\partial x_{n-k}} = 1,$$

and LSC implies that

$$Q(n+1, x_{n+1}) = \frac{\partial f_n}{\partial x_n} Q(n, x_n) + \frac{\partial f_n}{\partial x_{n-1}} Q(n-1, x_{n-1}) + \cdots + \frac{\partial f_n}{\partial x_{n-k}} Q(n-k, x_{n-k}),$$

if we compare the above two equations, we find that

$$Q(n, x_n) = c,$$

where c is a constant. □

Now, to solve Eq.(5.47) using Lie symmetry , an invariant v_n can be found by using

$$\frac{dx_n}{c} = \frac{dx_{n-1}}{c} = \cdots = \frac{dx_{n-k}}{c} = \frac{dv}{0},$$

taking the i^{th} and j^{th} invariants for any $i, j = n-k, \cdots, n$.

We solve,

$$dx_i = dx_j,$$

we get

$$x_i = x_j + k_{ij}, \qquad k_{ij} \in \mathbb{R}.$$

And then we solve for $i = n-k, \cdots, n$,

$$\frac{dx_i}{c} = \frac{dv}{0},$$

$$\int dv = \int 0,$$

Then, the invarient

$$v = constant = \psi(k_{n,n-1}, \cdots, k_{n,n-k}, k_{n-1,n}, k_{n-1,n-2}, \cdots, k_{n-1,n-k}, \cdots, k_{n-k,n-k+1}),$$

take $\psi(k_{n,n-1}, \cdots, k_{n,n-k}, \cdots, k_{n-1,n-2}, \cdots, k_{n-1,n-k}, \cdots, k_{n-k,n-k+1}) = k_{n,n-1}$, this gives

$$v_n = x_n - x_{n-1}.$$

Also this invarient satisfies

$$Iv_n = 0,$$

since

$$\begin{aligned} Iv_n &= Q(n, x_n)\frac{\partial(x_n - x_{n-1})}{\partial x_n} + Q(n-1, x_{n-1})\frac{\partial(x_n - x_{n-1})}{\partial x_{n-1}} + \cdots \\ &\quad + Q(n-k, x_{n-k})\frac{\partial(x_n - x_{n-1})}{\partial x_{n-k}} \\ &= c(1) + c(-1) + \cdots + c(0) \\ &= 0. \end{aligned}$$

The canonical coordinate in this case is given by

$$s_n = \int \frac{dx_n}{c} = \frac{x_n}{c} + c_1, \quad c, c_1 \in \mathbb{R},$$

and we choose a canonical coordinate $s_n = x_n$, and take $v_n = x_n - x_{n-1}$. It follows that

$$s_{n+1} - s_n = x_{n+1} - x_n = v_{n+1} = f_n(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}) - x_n,$$

since $\forall n$, f_n is $AH1$,

$$\begin{aligned} v_{n+1} &= f_n(x_n - x_n, x_{n-1} - x_n, x_{n-2} - x_n, \ldots, x_{n-k} - x_n) \\ &= f_n(0, -(x_n - x_{n-1}), -(x_{n-1} - x_{n-2}) - (x_n - x_{n-1}), \ldots, -(x_{n-k+1} - x_{n-k}) \\ &\quad - \cdots - (x_n - x_{n-1})) \\ &= f_n(0, -v_n, -v_{n-1} - v_n, \ldots, -v_{n-k+1} - \cdots - v_{n-1} - v_n). \end{aligned} \tag{5.48}$$

Equation (5.48) is of order k, we solve it for v_n, to find the solution of the original equation (5.47). We use the canonical coordinate $s_n = x_n$, and

$$s_{n+1} = s_n + v_{n+1},$$

which is a first order difference equation, whose solution is given by

$$s_n = s_0 + \sum_{i=1}^{n} v_i,$$

therefore,

$$x_n = x_0 + \sum_{i=1}^{n} v_i, \quad n = 1, 2, 3, \ldots.$$

5.4 Introduction To Stability

In this section, we review some relations and results which will be useful in our investigation.

Definition 5.5. *[3] Let I be some interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of inital conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), n = 0, 1, \ldots \tag{5.49}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 5.6. *[3] A point $\bar{x} \in I$ is called an equilibrium point of the Eq.(5.49) if*

$$\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n = 0, 1, \ldots,$ is a solution of Eq.(5.49), or equivalently, $\bar{x}$ is fixed point of f.

Definition 5.7. *(Stability)[3]*

- *The equilibrium point $\bar{x}$ of Eq.(5.49) is locally stable if for $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, with*

  $$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta,$$

  we have

  $$|x_n - \bar{x}| < \epsilon, \ \forall n \geq -k.$$

- *The equilibrium point $\bar{x}$ of Eq.(5.49) is locally asymptotically stable if $\bar{x}$ is locally stable of Eq.(5.49) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, with*

  $$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma,$$

  we have

  $$\lim_{n \to \infty} x_n = \bar{x}.$$

- *The equilibrium point $\bar{x}$ of Eq.(5.49) is global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$, we have*

  $$\lim_{n \to \infty} x_n = \bar{x}.$$

- *The equilibrium point $\bar{x}$ of Eq.(5.49) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also global attractor of Eq.(5.49).*

- *The equilibrium point $\bar{x}$ of Eq.(5.49) is unstable if $\bar{x}$ is not locally stable.*

Definition 5.8. *[3] The linearized equation of Eq.(5.49) about the equilibrium point $\bar{x}$ is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 5.9. *(Linearized Stability)[3] Assume that $p_i \in \mathbb{R}$, $i = 1, 2, \dots$ and $k = \{0, 1, \dots\}$. Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, n = 0, 1, \dots$$

5.5 Multiplicative Homogeneous Equations of Second Order

Let

$$x_{n+1} = f(x_n, x_{n-1}), n = 0, 1, \dots \tag{5.50}$$

be a second order $MH1$. Therefore, this equation is equivalent to

$$r_{n+1} = f(1, r_n^{-1}). \tag{5.51}$$

It is convenient in what follows to define the mapping

$$g(r) = f(1, r^{-1}),$$

so that Eq.(5.51) can be written as $r_{n+1} = g(r_n)$. Here, we focus on positive solutions. Let f be a positive function, and assume that the continuous map $g : (0, \infty) \to (0, \infty)$ has a unique fixed point $\bar{r}$, so that $\bar{r}$ is an equilibrium of Eq.(5.51). Then the ray $\{(x, \bar{r}x) : x \in (0, \infty)\}$ or $\bar{r}x$ for short is an invariant set of Eq.(5.50) in the state−space $(0, \infty)^2$. Since if (x_{-1}, x_0) is a point on this ray so that $x_0 = \bar{r}x_{-1}$, then $r_0 = \bar{r}$ and thus $r_1 = \bar{r}$, since

$$r_1 = g(r_0) = g(\bar{r}) = \bar{r},$$

therefore, $x_1 = \bar{r}x_0$; i.e.(x_0, x_1) is on $\bar{r}x$. By induction, the state−space orbit (x_{n-1}, x_n) is on the invariant ray for all n.
So we have

$$x_n = r_n x_{n-1} = r_n r_{n-1} x_{n-2} = \cdots = r_n r_{n-1} \ldots r_1 x_0,$$

and thus

$$x_n = (\bar{r})^n x_0.$$

We have three cases according to the value of $\bar{r}$,

- If $\bar{r} < 1$, then every orbit of Eq.(5.50) starting on $\bar{r}x$ will converge monotonically to 0 on $\bar{r}x$.
- If $\bar{r} = 1$, then every orbit in $\bar{r}x$ is stationary (a point).
- If $\bar{r} > 1$, then every orbit of Eq.(5.50) starting on $\bar{r}x$ will goes to ∞ monotonically.

We see that the invariant ray $\bar{r}x$ is analogous to a fixed point for Eq.(5.50), in the sense that by taking the quotient of $(0, \infty)^2$ modulo $\bar{r}x$, Eq.(5.50) is transformed into a topological conjugate of Eq.(5.51), and the ray $\bar{r}x$ into the point $\bar{r}$; on the space of rays through the origion. We conclude to,

Theorem 5.10. [9](Solutions on the invariant ray)

- If $\bar{r} < 1$, then the solutions of Eq.(5.51) on the invariant ray $\bar{r}x$ converger to 0 monotonically.

- If $\bar{r} = 1$, then the solutions of Eq.(5.51) on the invariant ray $\bar{r}x$ are stationary or contant solutions.
- If $\bar{r} > 1$, then the solutions of Eq.(5.51) on the invariant ray $\bar{r}x$ converges to ∞ eventually monotonically.

Theorem 5.11. [9]

- Let 1 be globally attracting. Then,
  - If $\bar{r} < 1$, then every positve solution of Eq.(5.50) converges to 0 eventually monotonically.
  - If $\bar{r} > 1$, then every positve solution of Eq.(5.50) converges to ∞ eventually monotonically.
- Let for $r > 0$, $g(r) < r$. Then, every positve solution of Eq.(5.50) converges to 0 eventually monotonically.

Lemma 5.12. [9] Let x_n be a given sequence of real numbers. If there exists a sequence y_n of positive real numbers such that

$$| x_{n+1} - x_n | \leq y_n | x_n - x_{n-1} |, \qquad n = 0, 1, \cdots ,$$

and $\lim_{n \longrightarrow \infty} y_n = y < 1$, then x_n converges to a finite limit.

Theorem 5.13. [9] Let 1 be globally attracting for Eq.(5.51). If $0 < (\partial f / \partial x)(1, 1) < 2$, then every positive solution of Eq.(5.50) converges to a finite limit.

Proof. 1 is globally attracting for Eq.(5.51), that is g has a fixed point at 1, and $\lim_{n \longrightarrow \infty} r_n = \frac{x_n}{x_{n-1}} = 1$. Then, to use the above lemma we write

$$| x_{n+1} - x_n | = \left| \frac{f(x_n, x_{n-1}) - x_n}{x_n - x_{n-1}} \right| | x_n - x_{n-1} | = \left| \frac{f(\frac{x_n}{x_{n-1}}, 1) - \frac{x_n}{x_{n-1}}}{\frac{x_n}{x_{n-1}} - 1} \right| | x_n - x_{n-1} | .$$

Assume that

$$y_n = \left| \frac{f(\frac{x_n}{x_{n-1}}, 1) - \frac{x_n}{x_{n-1}}}{\frac{x_n}{x_{n-1}} - 1} \right|,$$

Clearly, y_n is positive, and we can find its limit by L'Hpspital's rule.

$$\lim_{n \longrightarrow \infty} y_n = \lim_{n \longrightarrow \infty} \left| \frac{f(\frac{x_n}{x_{n-1}}, 1) - \frac{x_n}{x_{n-1}}}{\frac{x_n}{x_{n-1}} - 1} \right| = \left| \frac{\partial f}{\partial x}(1, 1) - 1 \right|.$$

Now,

$$\left| \frac{\partial f}{\partial x}(1, 1) - 1 \right| < 1 \Leftrightarrow 0 < \frac{\partial f}{\partial x}(1, 1) < 2.$$

Therefore, by the above lemma x_n conveges to a finite limit. The proof is complete. □

The following example, illustrates the above theorem.

Example 5.14. Let the $MH1$ of order two

$$x_{n+1} = \frac{2x_{n-1}x_n}{x_{n-1} + x_n}, \tag{5.52}$$

in example (3.22), we find its solution and is given by

$$x_n = \frac{1}{\hat{c_1} + \hat{c_2}(-2)^{-n}}, \tag{5.53}$$

where $\hat{c_1}$ and $\hat{c_2} \in \mathbb{R}$, are not both zero.

Now, the reduced equation of Eq.(5.52) is given by

$$r_{n+1} = \frac{2}{r_n + 1}, \tag{5.54}$$

which is a first order non-linear difference equation, its fixed point are 1 and -2, we find them by solving the following equation

$$\bar{r} = \frac{2}{\bar{r} + 1} \Rightarrow \bar{r}^2 + \bar{r} - 2 = 0 \Rightarrow (\bar{r} + 2)(\bar{r} - 1) = 0 \Rightarrow \bar{r} = 1, -2.$$

To prove that 1 is a global attractor of Eq.(5.54), we need to find its general solution.

Eq.(5.54) which is a type two of Ricatti Equation, that is a non-linear first order difference equation that we can transform to a second order linear difference equation as follows,

[12] **Equations of general Riccati type 2:**

$$x_{n+1} = \frac{a(n)x_n + b(n)}{c(n)x_n + d(n)}, \tag{5.55}$$

where $c(n) \neq 0$, and $a(n)d(n) - b(n)c(n) \neq 0$ for all $n \geq 0$.

To solve it, we let

$$c(n)x_n + d(n) = \frac{z_{n+1}}{z_n},$$

then we substitute

$$x_n = \frac{z_{n+1}}{c(n)z_n} - \frac{d(n)}{c(n)},$$

into equation (5.55), we obtain

$$\left(\frac{z_{n+2}}{c(n+1)z_{n+1}} - \frac{d(n+1)}{c(n+1)}\right)\left(\frac{z_{n+1}}{z_n}\right) = a(n)\left(\frac{z_{n+1}}{c(n)z_n} - \frac{d(n)}{c(n)}\right) + b(n).$$

Multiply this equation by $c(n+1)z_n$, we get

$$z_{n+2}-d(n+1)z_{n+1}-a(n)\frac{c(n+1)z_{n+1}}{c(n)}+\left(\frac{a(n)d(n)c(n+1)}{c(n)}-b(n)c(n+1)\right)z_n = 0,$$

which is equivalent to

$$z_{n+2}-\left(d(n+1)-a(n)\frac{c(n+1)}{c(n)}\right)z_{n+1}+\left(\frac{a(n)d(n)c(n+1)}{c(n)}-b(n)c(n+1)\right)z_n = 0,$$

this equation is of the form

$$z_{n+2} + g_1(n)z_{n+1} + g_2(n)z_n = 0,$$

which is linear difference equation. Now to solve Eq.(5.54), we note that $a(n) = 0, b(n) = 2, c(n) = d(n) = 1$, therefore $c(n) \neq 0$ and $a(n)d(n) - b(n)c(n) = -2 \neq 0, \forall n \geq 0$. So we let

$$r_n + 1 = \frac{z_{n+1}}{z_n}, \tag{5.56}$$

we obtain

$$z_{n+2} - z_{n+1} - 2z_n = 0.$$

The characteristic equation is

$$r^{n+2} - r^{n+1} - 2r^n = 0,$$

which implies that

$$r^2 - r - 2 = 0,$$

so the characteristic roots are: $r = 2$ and $r = -1$ and the general solution is

$$z_n = c_1 2^n + c_2(-1)^n,$$

where c_1 and $c_2 \in \mathbb{R}$, are not both zero. From (5.56) we have

$$\begin{aligned} r_n &= \frac{c_1(2)^{n+1} + c_2(-1)^{n+1}}{c_1(2)^n + c_2(-1)^n} - 1 \\ &= \frac{2c_1(2)^n - c_2(-1)^n}{c_1(2)^n + c_2(-1)^n} - 1, \end{aligned}$$

where c_1 and $c_2 \in \mathbb{R}$, not both are zero. Taking the limit of the solution of r_n,

$$\lim_{n\longrightarrow\infty} r_n = \lim_{n\longrightarrow\infty} \frac{2c_1(2)^n - c_2(-1)^n}{c_1(2)^n + c_2(-1)^n} - 1,$$

divide the above limit by 2^n, we have

$$\lim_{n\longrightarrow\infty} r_n = \lim_{n\longrightarrow\infty} \frac{2c_1 - c_2(\frac{-1}{2})^n}{c_1 + c_2(\frac{-1}{2})^n} - 1 = \frac{2c_1 - 0}{c_1 + 0} - 1 = 2 - 1 = 1.$$

We conclude that 1 is a global attractor of Eq.(5.54).

And $0 < (\partial f/\partial x)(1,1) = \frac{1}{2} < 2$. Therefore, by the above theorem every positive solution of Eq.(5.52) converges to a finite limit. To see that, we find the limit of the solution of Eq.(5.52) that is in (5.53) as follows:

$$\lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} \frac{1}{\hat{c_1} + \hat{c_2}(-2)^{-n}} = \frac{1}{\hat{c_1}}.$$

6 ON DIFFERENCE EQUATION $X_{N+1} = \frac{X_N X_{N-K}}{AX_{N-K}+BX_{N-L}}$

In this chapter, we will study the dynamics of the $MH1$ difference equation of order $k+1$

$$x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k} + bx_{n-l}}, \tag{6.1}$$

we find that Eq.(6.1) has no fixed points if $a + b \neq 1$. Therefore, we take a special case of Eq.(6.1) with $a = b = 1$ which becomes

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k} + x_{n-l}}. \tag{6.2}$$

We find the exact solution of Eq(6.2) by using Theorem (5.3) and study its global behaviour according to its solution. And we will study the local and global behaviour of its reduced equation that is given by

$$r_{n+1} = \frac{1}{1 + r_{n-k+1}}.$$

Finally, we will present Matlab for their solutions.

6.1 Equilibrium Points Of The Difference equation $x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k}+bx_{n-l}}$

In this section, we find the condition on a and b in finding the equilibrium points of the difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k} + bx_{n-l}} \tag{6.3}$$

where the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0$, and a, b are arbitrary positive real numbers. The equilibrium points of Eq.(6.3) are given by the relation

$$\bar{x} = \frac{\bar{x}^2}{a\bar{x} + b\bar{x}} \Rightarrow \bar{x}^2(a+b) = \bar{x}^2 \Rightarrow \bar{x}^2(a+b-1) = 0,$$

If $a + b \neq 1$, then there is no positve equilibrium points of Eq.(6.3). And if $a+b = 1$, then the equilibrium points of Eq.(6.3) are $\bar{x} = (c, c, \cdots, c), c \in \mathbb{R}^*$.

6.2 Exact Solution of the Difference Equation $x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k}+bx_{n-l}}$

In this section, we find the exact solution of the $MH1$ difference equation

$$x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k} + bx_{n-l}}, \tag{6.4}$$

of order $k+1$, with $k > l$ by order reduction method for $HD1$. Specifically we choose $k = l+1$; to find an explicit formula of the solution of equation

$$x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k} + bx_{n-l}},$$

with $x_{-k}, x_{-k+1}, \cdots, x_0$ initial values.Now, if we let $l = k-1$ in Eq.(6.4) we have

$$x_{n+1} = \frac{x_n x_{n-k}}{ax_{n-k} + bx_{n-k+1}} = f(x_n, \cdots, x_{n-k+1}, x_{n-k}), \tag{6.5}$$

and the reduced equation is given by,

$$\begin{aligned} r_{n+1} &= f(1, \cdots, r_n^{-1} r_{n-1}^{-1} \cdots r_{n-k+2}^{-1}, r_n^{-1} r_{n-1}^{-1} \cdots r_{n-k+2}^{-1} r_{n-k+1}^{-1}) \\ &= \frac{1}{a + b r_{n-k+1}}, \end{aligned} \tag{6.6}$$

Eq.(6.6) can be written as

$$r_{n+k} = \frac{1}{a + b r_n}, \tag{6.7}$$

which is an equation of order k that can be solved recursively. Let $r_0, r_1, \cdots, r_{k-1}$ be given and $a = b = 1$, then

$$\begin{aligned} &n = 0, \quad r_k = \frac{1}{1 + r_0}, \\ &n = 1, \quad r_{k+1} = \frac{1}{1 + r_1}, \\ &\vdots \\ &n = k - 1, \quad r_{2k-1} = \frac{1}{1 + r_{k-1}}, \end{aligned}$$

$$\begin{aligned} &n = k, \quad r_{2k} = \frac{1}{1 + r_k} = \frac{1 + r_0}{2 + r_0}, \\ &n = k + 1, \quad r_{2k+1} = \frac{1}{1 + r_{k+1}} = \frac{1 + r_1}{2 + r_1}, \\ &\vdots \\ &n = 2k - 1, \quad r_{3k-1} = \frac{1}{1 + r_{2k-1}} = \frac{1 + r_{k-1}}{2 + r_{k-1}}, \end{aligned}$$

$$\begin{aligned} &n = 2k, \quad r_{3k} = \frac{1}{1 + r_{2k}} = \frac{2 + r_0}{3 + 2r_0}, \\ &n = 2k + 1, \quad r_{3k+1} = \frac{1}{1 + r_{2k+1}} = \frac{2 + r_1}{3 + 2r_1}, \\ &\vdots \\ &n = 3k - 1, \quad r_{4k-1} = \frac{1}{1 + r_{3k-1}} = \frac{2 + r_{k-1}}{3 + 2r_{k-1}}, \end{aligned}$$

Let $f(n)$ be the Fibonacci numbers which satisfy the recurrence relation

$$f(n) = f(n-1) + f(n-2); \quad n \geq 2,$$

where $f(0) = 0$ and $f(1) = 1$. This is a *second* order linear difference equation whose general solution is given by

$$f(n) = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{6.8}$$

We note that

$$\begin{aligned}
r_k &= \frac{1+0r_0}{1+1r_0} = \frac{f(1)+f(0)r_0}{f(2)+f(1)r_0} \\
r_{2k} &= \frac{1+r_0}{2+1r_0} = \frac{f(2)+f(1)r_0}{f(3)+f(2)r_0} \\
r_{3k} &= \frac{2+1r_0}{3+2r_0} = \frac{f(3)+f(2)r_0}{f(4)+f(3)r_0} \\
&\vdots
\end{aligned}$$

we conclude that the formula of the solution with respect to r_0 is

$$r_n = r_{mk} = \frac{f(m)+f(m-1)r_0}{f(m+1)+f(m)r_0}, \quad m = 1, 2, 3, \cdots \tag{6.9}$$

and

$$\begin{aligned}
r_{k+1} &= \frac{1+0r_1}{1+1r_1} = \frac{f(1)+f(0)r_1}{f(2)+f(1)r_1} \\
r_{2k+1} &= \frac{1+r_1}{2+1r_1} = \frac{f(2)+f(1)r_1}{f(3)+f(2)r_1} \\
r_{3k+1} &= \frac{2+1r_1}{3+2r_1} = \frac{f(3)+f(1)r_1}{f(4)+f(3)r_1} \\
&\vdots
\end{aligned}$$

also we conclude the formula of the solution with respect to r_1 is

$$r_n = r_{mk+1} = \frac{f(m)+f(m-1)r_1}{f(m+1)+f(m)r_1}, \quad m = 1, 2, 3, \cdots \tag{6.10}$$

and if we do the same iterations and calculations as above we will reach to the last r_{k-1}, and we conclude that

$$r_n = r_{mk+k-1} = \frac{f(m) + f(m-1)r_{k-1}}{f(m+1) + f(m)r_{k-1}}, \quad m = 1, 2, 3, \cdots . \tag{6.11}$$

Therefore, the general solution of Eq.(6.6) is

$$r_n = r_{mk+t} = \frac{f(m) + f(m-1)r_t}{f(m+1) + f(m)r_t}, \qquad m = 1, 2, 3, \cdots, \forall t = 0, 1, \cdots, k-1. \tag{6.12}$$

We can write Eq.(6.12) as follows

$$r_n = \begin{cases} \frac{f(m)+f(m-1)r_0}{f(m+1)+f(m)r_0}; n = mk, m = 1, 2, 3, \cdots . \\ \\ \frac{f(m)+f(m-1)r_1}{f(m+1)+f(m)r_1}; n = mk+1, m = 1, 2, 3, \cdots . \\ \\ \quad \vdots \\ \\ \frac{f(m)+f(m-1)r_{k-1}}{f(m+1)+f(m)r_{k-1}}; n = mk+k-1, m = 1, 2, 3, \cdots . \end{cases}$$

Lemma 6.1. *The general solution of the k^{th} order difference equation*

$$r_{n+k} = \frac{1}{1 + r_n}$$

is given by

$$r_n = r_{mk+t} = \frac{f(m) + f(m-1)r_t}{f(m+1) + f(m)r_t}$$

where $m = 1, 2, 3, \cdots \ \forall t = 0, 1, \cdots, k-1$.

Proof. By induction.

Firstly, it's true for $m = 1, \forall t = 0, 1, \cdots, k-1$. Since

$$r_n = r_{k+t} = \frac{f(1) + f(0)r_t}{f(2) + f(1)r_t} = \frac{1}{1 + r_t}.$$

Suppose it's true for $m-1, \forall t = 0, 1, \cdots, k-1$,

$$r_{(m-1)k+t} = \frac{f(m-1) + f(m-2)r_t}{f(m) + f(m-1)r_t}.$$

Now, we need to prove it for m,

$$r_{mk+t} = \frac{1}{1 + r_{(mk+t)-k}} = \frac{1}{1 + r_{(m-1)k+t}},$$

we substitute $r_{(m-1)k+t}$ from our assumption to get,

$$\begin{aligned} r_{mk+t} &= \frac{1}{1 + \frac{f(m-1)+f(m-2)r_t}{f(m)+f(m-1)r_t}} \\ &= \frac{f(m) + f(m-1)r_t}{f(m) + f(m-1) + (f(m-1) + f(m-2))r_t}, \end{aligned}$$

since $f(m) = f(m-1) + f(m-2)$, we get

$$r_{mk+t} = \frac{f(m) + f(m-1)r_t}{f(m+1) + f(m)r_t}.$$

This proves our result. □

Now, the general solution of Eq.(6.4) is given by

$$x_n = x_0 \prod_{i=1}^{n} r_i = x_0 \prod_{i=1}^{n} \frac{f(i) + f(i-1)r_t}{f(i+1) + f(i)r_t}, \tag{6.13}$$

where $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are positve initial points of Eq.(6.4) and $r_t = x_t x_{t-1}^{-1}$, $\forall t = 0, 1, 2, \cdots, k-1$.

Theorem 6.2. *The general solution of the $k+1$ order difference equation*

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k} + x_{n-k+1}},$$

is given by

$$x_n = x_0 \prod_{i=1}^{n} \frac{f(i) + f(i-1)r_t}{f(i+1) + f(i)r_t},$$

where $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are positve initial points of Eq.(6.4) and $r_t = x_t x_{t-1}^{-1}$, $\forall t = 0, 1, 2, \cdots, k-1$.

Proof. The proof is following by order reduction theorem(4.5) and lemma(6.1). □

6.3 Local Stability of $r_n = \frac{1}{1+r_{n-k}}$

In this section, we study the local stability character of the positive equilibrium point of the difference equation

$$r_n = \frac{1}{1 + r_{n-k}}, \tag{6.14}$$

where the initial conditions $r_{-k}, r_{-k+1}, \cdots, r_0$ are arbitrary positive real numbers. The equilibrium points of Eq.(6.14) are given by the relation

$$\bar{r} = \frac{1}{1+\bar{r}} \Rightarrow \bar{r}^2 + \bar{r} - 1 = 0 \Rightarrow \bar{r} = \frac{-1 \pm \sqrt{5}}{2},$$

the positive equlibrium is

$$\bar{r} = \frac{-1+\sqrt{5}}{2}.$$

Let

$$g : (0, \infty) \to (0, \infty),$$

be a function defined by

$$g(u) = \frac{1}{1+u},$$

therefore,

$$\frac{dg}{du} = \frac{-1}{(1+u)^2}.$$

Then we see that

$$\frac{dg}{du}\left(\frac{-1+\sqrt{5}}{2}\right) = \frac{-4}{6+2\sqrt{5}} = \gamma.$$

Then the linearized equation of Eq.(6.14) about $\bar{r} = \frac{-1+\sqrt{5}}{2}$ is

$$y_n + \gamma y_{n-k} = 0.$$

Since

$$\gamma = |\frac{-4}{6+2\sqrt{5}}| < 1,$$

it follows from the linearized stability theorem (5.9) that the positive equilibrium point of Eq(6.14) is locally asymptotically stable.

Theorem 6.3. *The positive equilibrium point of the difference equation*

$$r_n = \frac{1}{1 + r_{n-k}}, \tag{6.15}$$

is locally asymptotically stable.

6.4 Global stability of $r_n = \frac{1}{1+r_{n-k}}$

In this section, we investigate the global behavior of the k^{th} order difference equation

$$r_n = \frac{1}{1 + r_{n-k}}, \tag{6.16}$$

using the explicit formula of it's solution.

Theorem 6.4. *Let $\{r_n\}_{n=-k}^{\infty}$ be a solution of Eq.(6.16). Then $\{r_n\}_{n=-k}^{\infty}$ converges to a finite limit.*

Proof.

$$\lim_{n\longrightarrow\infty} r_n = \lim_{m\longrightarrow\infty} r_{mk+t} = \lim_{m\longrightarrow\infty} \frac{f(m) + f(m-1)r_t}{f(m+1) + f(m)r_t},$$

dividing by $f(m)$ we get

$$\lim_{n\longrightarrow\infty} r_n = \lim_{m\longrightarrow\infty} \frac{1 + \frac{f(m-1)}{f(m)} r_t}{\frac{f(m+1)}{f(m)} + r_t},$$

since $\frac{f(n+1)}{f(n)}$ conveges to α as $n \longrightarrow \infty$. Therefore,

$$\lim_{n\longrightarrow\infty} r_n = \frac{1}{\alpha} = \frac{2}{1+\sqrt{5}}.$$

□

6.5 Global Behavior of $x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k}+x_{n-l}}$

In this section, we investigate the global behavior of

$$x_{n+1} = \frac{x_n x_{n-k}}{x_{n-k} + x_{n-l}}, \tag{6.17}$$

using explicit formula of its solution.

Theorem 6.5. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq.(6.17). If the solution of Eq.(6.16) $\{r_n\}_{n=-k+1}^{\infty}$ converges to $\frac{1}{\alpha}$, then $\{x_n\}_{n=-k}^{\infty}$ converges to 0.*

Proof. Let the solution of Eq.(6.17) is

$$x_n = x_0 \prod_{i=1}^{n} r_i,$$

where x_0 is an arbitrary positive real contant. From the previous theorem we have, as $n \to \infty$, $r_n \to \frac{1}{\alpha}$, and note that $\frac{1}{\alpha} < 1$ since $\alpha = \frac{1+\sqrt{5}}{2}$. It follows that for a given $0 < \frac{1}{\alpha} < \epsilon < 1$, there exists $i_0 \in \mathbb{N}$ such that $r_{i+1} < \epsilon$, $\forall$ $i \geq i_0$. Therfore,

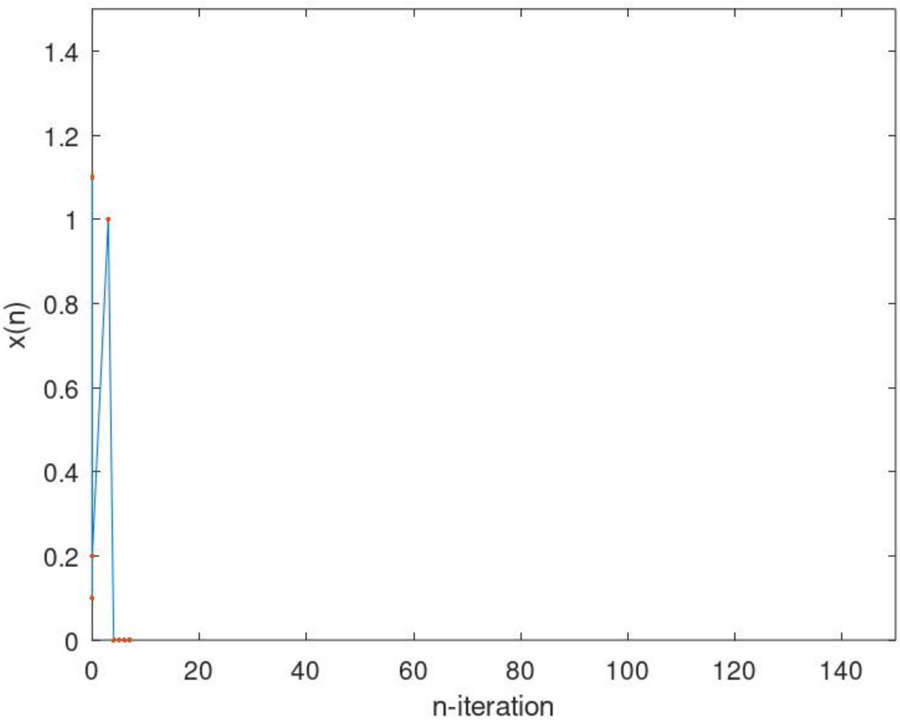
$$\begin{aligned} x_n &= x_0 \prod_{i=1}^{n} r_i \\ &= x_0 \prod_{i=1}^{i_0-1} r_i \prod_{i=i_0}^{n} r_i \\ &< x_0 \prod_{i=1}^{i_0-1} r_i \epsilon^{n-i_0} \end{aligned}$$

as $n \to \infty$, $x_n \to 0$. Therfore, $\{x_n\}_{n=-k}^{\infty}$ conerges to 0. □

6.6 Matlab code for chapter six

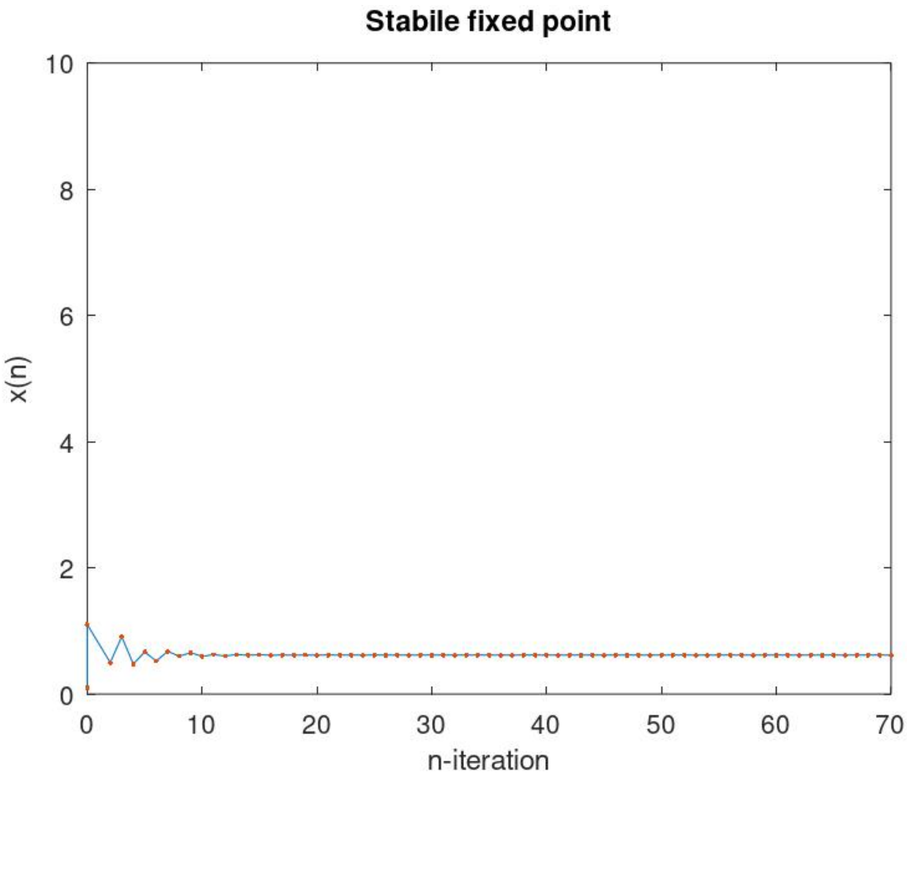
The behavior of the solution of $x_{n+1} = \frac{x_n x_{n-3}}{x_{n-3}+x_{n-2}}$.

```
n=150;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1; x(2)=1.1; x(3)=0.2; x(4)=1;
for i=4:n
t(i)=i-1;
x(i+1)=(x(i)*x(n-3))/(x(i-3)+x(i-2));
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis([0 150 0 1.5])
```



The behavior of the solution of $x_{n+1} = \frac{1}{1+x_{n-2}}$.

```
n=70;
x=zeros(n+1,1);
t=zeros(n+1,1); x(1)=0.1 ; x(2)=1.1; x(3)=0.5;
for i=3:n
t(i)=i-1;
x(i+1)=(1) / (1+x(i-2));
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis([0 70 0 10]), title('Stabile fixed point')
```



CONCLUSION

In this thesis we found that for a $HD1$ of order $k+1$ the characteristic function $Q(n, x_n)$ is x_n and c for $MH1$ and $AH1$ respectively. And insted of solvig them by Lie symmetry method, we can use reduction of order method that reduces the order of $HD1$ by *one*, and its enough to solve the reduced equation. Also, in Lie symmetry and reduction of order method we reached to the same solution of the original equation that is of order $k+1$.

We generlized the convergence of solutions of the original equation and its reduced equation. We found that if the the solution of the reduced equation converges to a finite limit. Then, its original equation converges to 0.

FUTURE WORK

It could be interesting to study another qualitative behaviour between the original equation and its reduced equation as periodicity of their solutions.

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